RELATIVE OUTER AUTOMORPHISMS OF FREE GROUPS

by

Erika Meucci

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STATEMENT OF DISSERTATION APPROVAL

The dissertation of Erika Meucci
has been approved by the following supervisory committee members:

Mladen Bestvina	, Chair	10/18/2011 Date Approved
Juan C. Souto	, Member	10/18/2011 Date Approved
Kenneth Bromberg	, Member	10/18/2011 Date Approved
Kevin Wortman	, Member	10/18/2011 Date Approved
Dragan Milicic	, Member	10/18/2011 Date Approved

and by Peter Trapa, Chair of
the Department of Mathematics,

and by Charles A. Wight, Dean of The Graduate School.

ABSTRACT

The study of automorphism groups of free groups is old, but the geometric approach to these groups is relatively new. Outer space was introduced in 1986 by Culler and Vogtmann as a tool for studying the group $\mathrm{Out}(F_n)$ of outer automorphisms of a finitely-generated free group. This work is focused on special subgroups of $\mathrm{Out}(F_n)$ called relative outer automorphisms groups. Let $A_1, \ldots, A_k$ be a system of free factors of F_n. The group of relative automorphisms $\mathrm{Aut}(F_n; A_1, \ldots, A_k)$ is the group given by the automorphisms of F_n that restricted to each A_i are conjugations by elements in F_n. The group of relative outer automorphisms is denoted by $\mathrm{Out}(F_n; A_1, \ldots, A_k)$ and defined as $\mathrm{Aut}(F_n; A_1, \ldots, A_k)/\mathrm{Inn}(F_n)$, where $\mathrm{Inn}(F_n)$ is the normal subgroup of $\mathrm{Aut}(F_n)$ given by all the inner automorphisms.

First, we define the relative outer space on which a relative outer automorphism group of a free group acts properly discontinuously and we compute the virtual cohomological dimension of relative outer automorphism groups of a free group. Then we introduce another space, the modified relative outer space, and we analyze its geometry and its dynamics. As a consequence, the Contracting Geodesics Theorem follows. This powerful theorem and an induction on the free factor system are the ingredients in the proof of the main application: every embedding of a lattice in $\mathrm{Out}(F_n)$ has finite image.

To my beloved mother Giuliana.

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CHAPTER 1

INTRODUCTION

The study of automorphism groups of a free group is old and important contributions were made by Jakob Nielsen (starting in 1915) and by J. H. C. Whitehead (starting in the 1930's). However, the geometric approach to these groups is relatively new. Outer space was introduced in 1986 by Culler and Vogtmann as a tool for studying the group $\mathrm{Out}(F_n)$ of outer automorphisms of the finitely-generated free group on n letters F_n. The basic idea of outer space is that the points in this space correspond to finite graphs with fundamental group isomorphic to F_n.

In [22], Culler and Vogtmann proved that outer space is a $(3n-4)$-dimensional simplicial complex with missing faces and $\mathrm{Out}(F_n)$ acts on this space with finite stabilizers. Moreover, they proved that outer space is contractible and as a consequence, they computed the virtual cohomological dimension of $\mathrm{Out}(F_n)$.

Outer space was born from an analogy with Teichmüller space. Given a surface S, the Teichmüller space of S is the space of marked hyperbolic metrics on S. The group that acts on this space is the mapping class group defined as the quotient of the group of orientation preserving homeomorphisms of S by the group of homeomorphisms of S which are homotopic to the identity. The elements of this group are called mapping classes. Thurston classified the mapping classes in three types: reducible, periodic, and pseudo-Anosov. Moreover, Thurston studied the mapping class group of a surface using the dynamics of its action on Teichmüller space. We would like to study $\mathrm{Out}(F_n)$ via the dynamics of its action on outer space. In [13], Bestvina and Handel introduced train tracks for graphs inspired by the work of Thurston for surfaces. Moreover, Bestvina, Handel, and Feighn in [8], and Lustig, introduced the notion of laminations on a free group as an analog of laminations in Thurston's theory.

This work is focused on special subgroups of $\mathrm{Out}(F_n)$ called relative outer automorphisms groups. Let $A_1, \ldots, A_k$ be a system of free factors of F_n. The group of relative automorphisms $\mathrm{Aut}(F_n; A_1, \ldots, A_k)$ is the group given by the automorphisms of F_n that restricted to each A_i are conjugations by elements in F_n. The group of relative outer automorphisms is defined as $\mathrm{Out}(F_n; A_1, \ldots, A_k) = \mathrm{Aut}(F_n; A_1, \ldots, A_k)/\mathrm{Inn}(F_n)$, where $\mathrm{Inn}(F_n)$ is the normal subgroup of $\mathrm{Aut}(F_n)$ given by all the inner automorphisms.

The main goal of this work is the proof of Theorem 215: if Γ is an irreducible lattice in a connected semisimple Lie group of real rank at least 2, then every homomorphism $\Gamma \to \mathrm{Out}(F_n)$ has finite image.

In Chapter 2 we will define the relative outer space on which a relative outer automorphism group of a free group acts properly discontinuously. The main results in this chapter are the contractibility of the relative outer space and the computation of the virtual cohomological dimension of relative outer automorphism groups of a free group.

We begin Chapter 3 modifying the definition of the relative outer space. The goal of the chapter is studying the geometry of this "modified" relative outer space. In particular, we will introduce the Lipschitz (non-symmetric) metric in the modified relative outer space and we will prove the existence of train track maps. Moreover, we will classify the elements of $\mathrm{Out}(F_n; A_1, \ldots, A_k)$ in three types, as in the case of the mapping class group, using the proof given by Bers and the approach in [4].

Chapter 4 is entirely dedicated to the dynamics of the modified relative outer space. First, we will introduce stable and unstable laminations. Then we will study the behavior of the stabilizer of a stable lamination proving that for each fully irreducible relative outer automorphism, the stabilizer of the stable lamination associated to this automorphism modulo the kernel of the action is virtually cyclic. Finally, we will prove that an irreducible relative outer automorphism with irreducible powers acts on the compactification of the modified relative outer space with north-south dynamics.

Chapter 5 contains the proof that the axes determined by fully irreducible relative outer automorphisms are strongly contracting geodesics. This is a relative version of the analog result for fully irreducible outer automorphisms in [1].

Finally, in Chapter 6 we will see some applications of the theory developed in the previous chapters. There are three applications. The first application is a Tits alternative for the groups of relative outer automorphisms of free groups modulo the kernel of the action. The second and most important application is a proof of Theorem 215. The last application is a study of axes in the Cayley graph of a relative outer automorphism group of a free group modulo the kernel of the action of this group onto the modified relative outer space.

CHAPTER 2

RELATIVE OUTER SPACE

In this chapter first we define the relative outer space on which a relative outer automorphism group of a free group acts properly discontinuously. Then we prove that this space is contractible generalizing [18]. Finally, we compute the virtual cohomological dimension of relative outer automorphism groups of a free group. An upper bound and a lower bound of this invariant were found in a particular case by Jensen and Wahl in [33], but in this chapter we compute the actual virtual cohomological dimension of a bigger class of groups.

2.1 $\mathrm{Out}(F_n; \mathcal{A})$ and $\mathrm{CV}_n(\mathcal{A})$

Let F_n denote the free group on n generators. We consider the group of automorphisms of F_n, denoted by $\mathrm{Aut}(F_n)$, and the group of outer automorphisms

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n),$$

where $\mathrm{Inn}(F_n)$ is the normal subgroup of $\mathrm{Aut}(F_n)$ given by all the inner automorphisms. Culler and Vogtmann introduced a space CV_n on which the group $\mathrm{Out}(F_n)$ acts with finite stabilizers and proved that CV_n is contractible. That space CV_n is called *outer space*. See [43] and [3] for a survey on $\mathrm{Out}(F_n)$ and CV_n.

Let $A_1, \ldots, A_k$ be a system of free factors of F_n, i.e., there exists $B < F_n$ such that $F_n = A_1 * \cdots * A_k * B$. We define the group of relative (to $A_1, \ldots, A_k$) automorphisms $\mathrm{Aut}(F_n; A_1, \ldots, A_k)$ given by the elements $f \in \mathrm{Aut}(F_n)$ such that f restricted to each A_i is a conjugation by an element in F_n.

Obviously, $\mathrm{Aut}(F_n) > \mathrm{Aut}(F_n; A_1, \ldots, A_k) \triangleright \mathrm{Inn}(F_n)$. We define also the group of relative (to $A_1, \ldots, A_k$) outer automorphisms:

$$\mathrm{Out}(F_n; A_1, \ldots, A_k) = \mathrm{Aut}(F_n; A_1, \ldots, A_k)/\mathrm{Inn}(F_n) < \mathrm{Out}(F_n).$$

The goal of this chapter is to define the relative outer space $\mathrm{CV}_n(A_1, \ldots, A_k)$ on which $\mathrm{Out}(F_n; A_1, \ldots, A_k)$ acts with finite stabilizers and prove that $\mathrm{CV}_n(A_1, \ldots, A_k)$ is contractible. Moreover, we will compute the virtual cohomological dimension of the group $\mathrm{Out}(F_n; A_1, \ldots, A_k)$.

Consider $A_i =< y_1^i, \dots, y_{s(i)}^i >$ and $F_n =< y_1^1, \dots, y_{s(k)}^k, x_1, \dots, x_{n-\sum_{i=1}^k s(i)} >$. By a graph, we mean a connected 1-dimensional CW complex. Let the *relative rose* $R_n(A_1, \dots, A_k)$ be a graph obtained by a wedge of $n-\sum_{i=1}^k s(i)$ circles attaching $\sum_{i=1}^k s(i)$ circles $C_1^1, \dots, C_{s(k)}^k$ on k stems (Figure 2.1). The edges are denoted by $C_1^1, \dots, C_{s(k)}^k, f_1, \dots, f_k, e_1, \dots, e_{n-\sum_{i=1}^k s(i)}$. Moreover,

$$\pi_1(R_n(A_1, \dots, A_k), v) \cong F_n =< y_1^1, \dots, y_{s(k)}^k, x_1, \dots, x_{n-\sum_{i=1}^k s(i)} >,$$

where v is the central vertex in $R_n(A_1, \dots, A_k)$ (see Figure 2.1), by declaring y_i^j to be the homotopy class of C_i^j and x_i to be the homotopy class of the loop e_i.

Let $(R_n(A_1, \dots, A_k), \underline{k})$ be the graph $R_n(A_1, \dots, A_k)$ equipped with inclusions $k_j : \bigvee_{i=1}^{s(j)} S^1 \to R_n(A_1, \dots, A_k)$ that identifies $\bigvee_{i=1}^{s(j)} S^1$ with $\bigvee_{i=1}^{s(j)} C_i^j$, for all $j = 1, \dots, k$.

Definition 1. Let Γ be a graph of rank n with vertices of valence at least 3, equipped with embeddings $l_j : \bigvee_{i=1}^{s(j)} S^1 \to \Gamma$ for $j = 1, \dots, k$. We call $\mathbb{B}_j = l_j(\bigvee_{i=1}^{s(j)} S^1)$ *wedge cycle*. The *dual graph* of the $\mathbb{B}_j$'s is the graph with one vertex for each wedge cycle, one vertex w for each intersection between two or more wedge cycles and edges between w and vertices corresponding to the wedge cycles meeting in w.

Definition 2. An $(A_1, \dots, A_k, n)$*-graph* $(\Gamma, \underline{l})$ is a finite graph Γ of rank n with vertices of valence at least 3, with possible separating edges, equipped with embeddings $l_j : \bigvee_{i=1}^{s(j)} S^1 \to \Gamma$ for $j = 1, \dots, k$, such that any two $\mathbb{B}_j$ intersect in at most a point and the dual graph of the $\mathbb{B}_j$'s is a forest.

Notation 3. We will denote by $\mathcal{A}$ the set of free factors $A_1, \dots, A_k$.

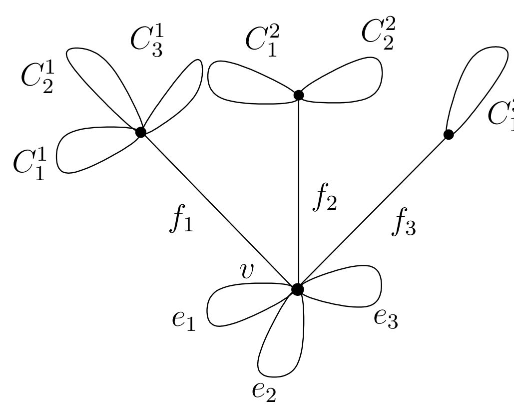

Figure 2.1. The relative rose $R_9(A_1, A_2, A_3)$.

Example 4. Consider the graph in Figure 2.2. Each loop of the same color is a wedge cycle. That graph is not an $(A_1, A_2, A_3, 4)$-graph because the dual graph of the $\mathbb{B}_j$'s is a circle (see Figure 2.3).

Definition 5. A *marked* $(\mathcal{A}, n)$*-graph* (Γ, ϕ) is a graph Γ of rank n equipped with a homotopy equivalence $\phi : R_n(\mathcal{A}) \to \Gamma$ such that $(\Gamma, \phi \circ \underline{k})$ is an $(\mathcal{A}, n)$-graph. The map ϕ is called the *marking*.

The marking induces an isomorphism $\phi_* : F_n \to \pi_1(\Gamma, \phi(v))$.

Definition 6. A *marked metric* $(\mathcal{A}, n)$*-graph* (Γ, ϕ) is a marked graph (Γ, ϕ) such that each edge e in Γ has positive real length $l(e)$.

Definition 7. The *relative outer space* $\mathrm{CV}_n(A_1, \ldots, A_k)$ (or $\mathrm{CV}_n(\mathcal{A})$) is the space of equivalence classes of marked metric $(\mathcal{A}, n)$-graphs where

1. the sum of all lengths of the edges in $\Gamma \setminus \{\phi(C_1^1), \ldots, \phi(C_{s(k)}^k)\}$ is 1 (we say that the relative volume 1) and
$$\sum_{e \subset \phi(C_i^j)} l(e) = 1 \quad \forall\, i, j;$$
2. $(\Gamma_1, \phi_1) \sim (\Gamma_2, \phi_2)$ if there is an isometry $h : \Gamma_1 \to \Gamma_2$ with h such that $h \circ \phi_1(C_i^j) = \phi_2(C_i^j)$, $\forall\, i, j$, and $h \circ \phi_1$ is homotopic to ϕ_2 rel. $\bigcup_{i,j} C_i^j$.

We will usually denote a point in $\mathrm{CV}_n(\mathcal{A})$ by (Γ, ϕ). There is a natural right action of $\mathrm{Out}(F_n; \mathcal{A})$ by homeomorphisms of $\mathrm{CV}_n(\mathcal{A})$: let $X = (\Gamma, \phi) \in \mathrm{CV}_n(\mathcal{A})$, let Ψ be a relative

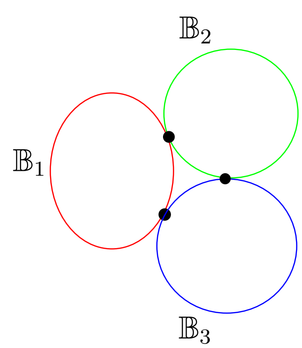

Figure 2.2. Example of a graph that is not an $(A_1, A_2, A_3, 4)$-graph.

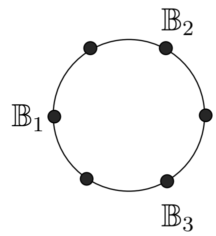

Figure 2.3. The dual graph of the graph in Figure 2.2.

outer automorphism and consider a map $\psi : R_n(\mathcal{A}) \to R_n(\mathcal{A})$ such that $[\psi_*] = \Psi$ and which is the identity on $\bigcup_{i,j} C_i^j$. Define

$$X \cdot \Psi = (\Gamma, \phi) \cdot \Psi = (\Gamma, \phi \circ \psi).$$

We can define a topology on $\mathrm{CV}_n(\mathcal{A})$ by varying the lengths of the edges exactly as for outer space (see [31] for the definition in the case of outer space).

Because we suppose that the relative volume is 1 and the sum of the lengths of the edges in each cycle $\phi(C_i^j)$ is 1, a point $(\Gamma, \phi) \in \mathrm{CV}_n(\mathcal{A})$ is in the interior of a polysimplex that is the product of the simplices Δ_i^j obtained by varying the lengths of the edges in each cycle $\phi(C_i^j)$ and the simplex σ given by varying the length of the edges in $\Gamma \setminus \{\phi(C_1^1), \ldots, \phi(C_{s(k)}^k)\}$. Indeed, if Γ has $N_{i,j}$ edges in $\phi(C_i^j)$ of length $s_{i,j}^1, \ldots, s_{i,j}^{N_{i,j}}$ and N edges in $\Gamma \setminus \{\phi(C_1^1), \ldots, \phi(C_{s(k)}^k)\}$ of length $t_1, \ldots, t_N$, then

$$\begin{array}{ccc} 0 < s_{i,j}^k < 1, \forall\, i, j, k & \text{and} & \sum_{k=1}^{N_{i,j}} s_{i,j}^k = 1, \\ 0 < t_m < 1, \forall\, m & \text{and} & \sum_{m=1}^{N} t_m = 1. \end{array}$$

Let Δ_i^j be the open simplex determined by varying the $s_{i,j}$'s and σ be the open simplex obtained by varying the t's. Define

$$P\Delta = \Delta_1^1 \times \cdots \times \Delta_{s(k)}^k \times \sigma.$$

Changing the length of the edges in Γ gives the open polysimplex $P\Delta$.

Example 8. Consider $\mathrm{Out}(F_5; A_1, A_2)$, where $F_5 =< a, a', b, b', c >$, $A_1 =< a, a' >$ and $A_2 =< b, b' >$, and consider the point $(\Gamma, \phi) \in \mathrm{CV}_5(A_1, A_2)$ in Figure 2.4.

The open polysimplex given by varying the length of the edges of Γ is the open cube $\Delta_1^1 \times \Delta_2^1 \times \Delta_1^2 \times \Delta_2^2 \times \sigma \cong \Delta_2^1 \times \Delta_1^2 \times \sigma$ (see Figure 2.5).

Note that $\mathrm{Out}(F_n; \mathcal{A})$ acts properly and discontinuously on $\mathrm{CV}_n(\mathcal{A})$ and that the stabilizer of any marked $(\mathcal{A}, n)$-graph (Γ, ϕ) is isomorphic to the subgroup of isometries of Γ which fixes the wedge cycles, hence it is finite.

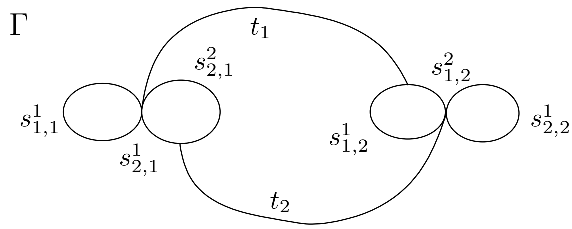

Figure 2.4. The point $(\Gamma, \phi) \in \mathrm{CV}_5(A_1, A_2)$ in Example 40.

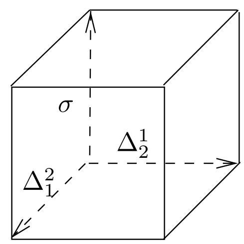

Figure 2.5. The open polysimplex $\Delta^1_2 \times \Delta^2_1 \times \sigma$ in Example 40.

For an $(\mathcal{A}, n)$-graph $(\Gamma, \underline{l})$, let $\widehat{\Gamma}$ be the graph of rank $n - \sum_{i=1}^{k} s(i)$ obtained from Γ by collapsing the wedge cycles $\mathbb{B}_1, \ldots, \mathbb{B}_k$ to points. Let e be an edge of Γ that does not define a loop in Γ or in $\widehat{\Gamma}$. Then the composition with the edge collapse col : $\Gamma \to \Gamma/e$ induces embeddings

$$l_j/e : \bigvee_{i=1}^{s(j)} S^1 \to \Gamma/e$$

such that $(\Gamma/e, \underline{l}/e)$ is again an $(\mathcal{A}, n)$-graph. By an edge collapse in an $(\mathcal{A}, n)$-graph, we will always mean the collapse satisfying the above hypothesis. For a marked $(\mathcal{A}, n)$-graph (Γ, ϕ), the marking of the collapsed graph is the composition $\mathrm{col} \circ \phi$. F is a forest in a graph Γ if it is a union of edges in Γ that does not contain any loop in Γ or in $\widehat{\Gamma}$. A forest collapse in an $(\mathcal{A}, n)$-graph $(\Gamma, \underline{l})$ is a sequence of edge collapses, where the edges that are collapsed are the edges in the forest. We denote the collapsed graph by $(\Gamma/F, \underline{l}/F)$. We define a poset structure on the set of marked $(\mathcal{A}, n)$-graphs by saying that $(\Gamma_1, \phi_1) \leq (\Gamma_2, \phi_2)$ if there is a forest F in Γ_2 such that $(\Gamma_2/F, \phi_2/F)$ is equivalent to (Γ_1, ϕ_1). We denote by $S_n(A_1, \ldots, A_k)$ (or $S_n(\mathcal{A})$) the geometric realization of that poset and we call it the *relative spine* of the relative outer space.

Example 9. Consider $n = 2$, $F_2 =< a, b >$ and $A =< a >$.

In that case, $\mathrm{Out}(F_2; A)$ is isomorphic to the infinite dihedral group D_∞. Indeed, if

$f \in \mathrm{Aut}(F_2)$ and $f(a) = a$, then $f(b) = a^n b^\varepsilon a^m$, where $n,\ m \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$. After conjugating by a power of a, we have $f(b) = a^N b$ or $f(b) = a^N b^{-1}$. The map

$$\begin{array}{ccc} \mathrm{Out}(F_2; A) & \to & \mathbb{Z}_2 \\ f & \mapsto & \varepsilon \end{array}$$

has kernel $\mathbb{Z}$, so we get

$$1 \to \mathbb{Z} \to \mathrm{Out}(F_2; A) \to \mathbb{Z}_2 \to 1,$$

i.e., $\mathrm{Out}(F_2; A) \cong \mathbb{Z} \rtimes \mathbb{Z}_2 \cong D_\infty$. The relative spine $S_2(A)$ is homeomorphic to the simplicial complex in Figure 2.6.

Remark 10. We can define the reduced relative spine as the subset of the geometric realization of the poset structure described previously, containing only the $(\mathcal{A}, n)$-graphs with no separating edges. In Example 9 the reduced relative spine is homeomorphic to a line.

Notice that $\mathrm{CV}_n(\mathcal{A})$ is not a polysimplicial complex, because some of the faces are missing, but the relative spine $S_n(\mathcal{A})$ is a simplicial complex.

Given a polysimplex $P\Delta$ we define its barycentric subdivision in the following way. Consider the centroid of each face or polyface (i.e., product of faces) of the polysimplex. Those will be the vertices of the barycentric subdivision and we will call them barycentric subdivision vertices (BSV). Now, for each $n > 0$ and n-face or n-polyface F, connect the centroid with each BSV in the $(n-1)$-faces (or polyfaces) of ∂F (see Figure 2.7).

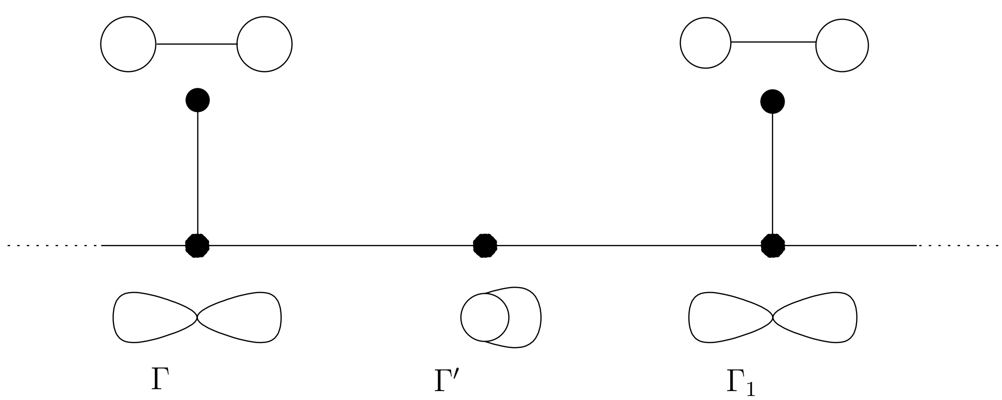

Figure 2.6. The spine $S_2(A)$ in Example 9 where (Γ, id) is the marked graph with identity marking, (Γ_1, ϕ_1) has the marking given by $\phi_1(a) = a$, $\phi_1(b) = ab$. In Γ and Γ_1 both the edges have length 1, while in Γ' the right edge has length 1 (it corresponds to $\phi'(e_1)$) and the other edges have length $\frac{1}{2}$.

Note that if Δ_j is a simplex face in $P\Delta$, then the restriction of the barycentric subdivision to $P\Delta_{|\Delta_j}$ is the (standard) barycentric subdivision of Δ_j. There is a natural embedding of $S_n(\mathcal{A})$ into $\mathrm{CV}_n(\mathcal{A})$ that sends each vertex to the centroid of the corresponding open polysimplex and each d-simplex to the convex hull of the corresponding centroids (see Figure 2.8 for an example of barycentric subdivision in $\mathrm{CV}_5(A_1, A_2)$ of Example 40).

$\mathrm{CV}_n(\mathcal{A})$ deformation retracts onto $S_n(\mathcal{A})$ in the following way. The vertices of $S_n(\mathcal{A})$ correspond to open polysimplices of $\mathrm{CV}_n(\mathcal{A})$ and a d-simplex is a chain of $d+1$ open polysimplices, each of which is a face of the next. By pushing within each open polysimplex of $\mathrm{CV}_n(\mathcal{A})$ away from the missing faces we have a deformation retraction from $\mathrm{CV}_n(\mathcal{A})$ to $S_n(\mathcal{A})$. In other words, $\mathrm{CV}_n(\mathcal{A})$ is the union of open polysimplices in a polysimplicial complex X. Note that $S_n(\mathcal{A})$ is the maximal full subcomplex of the barycentric subdivision of X that is disjoint from $X \setminus \mathrm{CV}_n(\mathcal{A})$. Collapsing every simplex in the barycentric subdivision of X to the face of the simplex contained in $S_n(\mathcal{A})$ gives a deformation retraction of $\mathrm{CV}_n(\mathcal{A})$ onto $S_n(\mathcal{A})$. The action of $\mathrm{Out}(F_n; \mathcal{A})$ extends to a simplicial action on the relative spine $S_n(\mathcal{A})$.

2.2 Contractibility of $\mathrm{CV}_n(\mathcal{A})$

The whole section is dedicated to the proof of the following theorem.

Theorem 11. The relative outer space $\mathrm{CV}_n(\mathcal{A})$ is contractible.

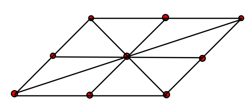

Figure 2.7. Barycentric subdivision of the polysimplex given by the product of two 1-simplices. The dots are the vertices of the barycentric subdivision.

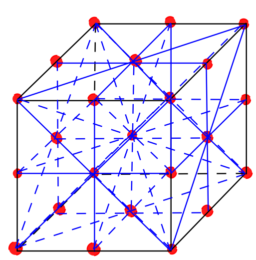

Figure 2.8. Barycentric subdivision of the polysimplex given in Example 40, where the red dots are the vertices of the simplices in $S_5(A_1, A_2)$.

Because there is a deformation retraction from the relative outer space $\mathrm{CV}_n(\mathcal{A})$ to its spine $S_n(\mathcal{A})$, it is enough to prove that the relative spine is contractible. First we prove that if $k = 1$, $S_n(\mathcal{A})$ is contractible. Recall from [31] that the spine S_n of the outer space CV_n is a poset of marked graphs of rank n, where the marking is given by a homotopy equivalence from the rose $R_n = \bigvee_1^n S^1$. S_n is contractible and admits an action of $\mathrm{Out}(F_n)$. Edge collapses induce a poset structure on S_n with minimal elements the *reduced* marked graphs, that is roses.

Let W be the set of conjugacy classes of elements in F_n. Let $w_1, \ldots, w_m$ be elements in W. Following [22], we define the function f_{w_i} from the set of roses to $\mathbb{R}$ by $f((R, \phi)) = nl(w_i)$, where l is the length function on F_n associated to R. The minset of f_{w_i} is

$$\mathrm{Minset}(f_{w_i}) = \bigcup_{f_{w_i} \text{ is min at } (R,\phi)} \mathrm{st}((R, \phi)),$$

where $\mathrm{st}(X)$ is the star of the point X in S_n.

Now, we consider the function f from the set of roses to $\mathbb{R}^m$, $f = (f_{w_1}, \ldots, f_{w_m})$ and we consider $\mathbb{R}^m$ equipped with the lexicographic order. Define Λ as the set of roses (R, ϕ) that are minimum in $f = (f_{w_1}, \ldots, f_{w_m})$ with respect to the lexicographic order. Let

$$\mathrm{Minset}(f) = \bigcup_{(R,\phi)\in\Lambda} \mathrm{st}((R, \phi)).$$

Remark 12. If (R_i, ϕ_i) are roses in S_n for $i = 1, 2$, $f((R_1, \phi_1)) \leq f((R_2, \phi_2))$, $(R_2, \phi_2) \in \Lambda$, then $(R_1, \phi_1) \in \Lambda$.

A useful lemma that we will need in the sequel is the Poset Lemma.

Theorem 13 (Poset Lemma)**.** Let X be a poset and $f : X \to X$ be a poset map with the property that $f(x) \leq x$ for all $x \in X$ (or $f(x) \geq x$ for all $x \in X$). Then $f(X)$ is a deformation retract of X.

See [38] for a proof of the Poset Lemma. Let (Γ, ϕ) be a marked graph in S_n and let v be a vertex of Γ. Formally, the notion of *ideal edges* is defined as in [22] in terms of partitions. We can think of an ideal edge γ at the vertex v as a partition of the set E_v of half edges of Γ terminating at v such that the blow-up Γ^γ in v is again in S_n, where Γ^γ is the graph obtained by pulling the half edges in γ away from v creating a new vertex $v(\gamma)$, a new edge γ that goes from $v(\gamma)$ to v and each half edge $e \subset \gamma$ is attached to $v(\gamma)$ instead of v. Note that the graph Γ can be reobtained by Γ^γ collapsing γ. An *ideal forest* in a reduced marked graph is a sequence of ideal edges. One can define a poset structure on the set of ideal forests of a rose (R, ϕ) such that the blowing up induces an isomorphism between that poset and the star of (R, ϕ) in S_n (see [22]). As for S_n, we can define ideal edges and ideal forests for $S_n(\mathcal{A})$.

Let A be a subset of E_v. We will denote by $\overline{A}$ the complement of A.

Definition 14. Two subsets A and B of E_v are *compatible* if one of the sets $A \cap B$, $\overline{A} \cap B$, $A \cap \overline{B}$, $\overline{A} \cap \overline{B}$ is empty.

The upper link of a marked $(\mathcal{A}, n)$-graph (Γ, ϕ) in $S_n(\mathcal{A})$ is a set of marked $(\mathcal{A}, n)$-graphs (Γ', ϕ') that collapse to (Γ, ϕ). Such marked $(\mathcal{A}, n)$-graphs are said to be obtained by blowing up vertices of Γ into trees. Notice that a set of ideal edges is compatible if it corresponds to a tree. Let $B(v)$ be the complex whose vertices are ideal edges at v and whose i-simplices are sets of $i + 1$ compatible ideal edges.

Definition 15. We say that an ideal edge γ at a vertex $v \in \Gamma$ is *legal* if $\Gamma^\gamma \in S_n(\mathcal{A})$. We denote the subcomplex of $B(v)$ spanned by legal ideal edges by $L(v)$.

Remark 16. An ideal edge is legal if and only if it separates at most one pair of half edges contained in a wedge cycle $\mathbb{B}_i$. Indeed, if it separates two pairs of half edges one in $\mathbb{B}_i$ and the other one in $\mathbb{B}_j$, then it blow ups to an edge in $\mathbb{B}_i \cap \mathbb{B}_j$ and that contradicts the definition of marked $(\mathcal{A}, n)$-graph.

We have the following remarkable result.

Theorem 17. Minset(f) is contractible.

The proof of that theorem follows from the following theorem in [22].

Theorem 18 (Culler-Vogtmann)**.** Let $W' = \{w_1, \ldots, w_m\}$, where $w_i \in W$ for all i. Then Minset(f) is a contractible subcomplex of S_n, the action is proper and the quotient Minset(f)/Stab(W') is finite.

An alternative proof is given in Section 4 of [33] (in the case of $G = \{1\}$, $\|\Lambda\| = \text{Minset}(f)$ and $E_n^G = S_n$).

Lemma 19 ([24])**.** If (Γ_1, ϕ_1) and (Γ_2, ϕ_2) are two marked graphs of rank n and $f : \Gamma_1 \to \Gamma_2$ is a map linear on edges such that the following diagram

$$\begin{tikzcd} \Gamma_1 \arrow[r, "f"] & \Gamma_2 \\ R_n \arrow[u, "\phi_1"] \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

commutes up to homotopy, then there is a subgraph of Γ_1 where the length of the edges are multiplied by the Lipschitz constant $\mathrm{Lip}(f)$ and the length of all the edges not in the subgraph are multiplied by a number strictly less than $\mathrm{Lip}(f)$.

Definition 20. Let (Γ_1, ϕ_1) and (Γ_2, ϕ_2) be two marked graphs of rank n. Given a map $f \sim \phi_2 \circ \phi_1^{-1}$ linear on edges, we denote by Γ_f the subgraph of Γ_1 whose edges are maximally stretched by $\mathrm{Lip}(f)$.

Definition 21. Let (Γ_1, ϕ_1) and (Γ_2, ϕ_2) be two marked graphs of rank n. A map $f \sim \phi_2 \circ \phi_1^{-1}$ linear on edges is *not optimal* if there is some vertex of Γ_f such that all the edges of Γ_f terminating at that vertex have f-image with a common terminal partial edge. Otherwise, f is called *optimal*.

A *turn* in (Γ, ϕ) is an unordered pair of oriented edges of Γ originating at a common vertex. A turn is *nondegenerate* if it is defined by distinct oriented edges. Otherwise, the turn is called *degenerate*. A map $f : \Gamma \to \Gamma$ induces a map Df from the set of oriented edges of Γ to itself by sending an oriented edge to the first oriented edge in its f-image as long as no edges are collapsed. We can think of Df as a sort of derivative. Df induces a map Tf on the set of turns in Γ. A turn is *illegal* with respect to f if its image under some iterate of Tf is degenerate. Otherwise, the turn is called *legal*. For properties of legal and illegal turns see [13] or [1]. Remember that

$$A_i =< y_1^i, \ldots, y_{s(i)}^i > \text{ and } F_n =< y_1^1, \ldots, y_{s(k)}^k, x_1, \ldots, x_{n-\sum_{i=1}^k s(i)} > .$$

Consider the function f given by $f = (f_1, f_2, \ldots, f_k)$, where

$$f_j = (f_{w_1^j}, \ldots, f_{w_{s(j)}^j}, f_{w_{1,2}^j}, \ldots, f_{w_{i,l}^j}, \ldots, f_{w_{i,\bar{l}}^j}, \ldots)$$

and

$$\begin{array}{rcll} w_i^j & = & y_i^j, & \text{for } i = 1, \ldots, s(k), \\ w_{i,l}^j & = & y_i^j y_l^j, & \text{for all } i < l,\ i, l = 1, \ldots, s(k), \\ w_{i,\bar{l}}^j & = & y_i^j \overline{y}_l^j, & \text{for all } i < l,\ i, l = 1, \ldots, s(k). \end{array}$$

Lemma 22. Consider $F_{s(j)} =< y_1^j, \ldots, y_{s(j)}^j >$. Then $\mathrm{Minset}(f_j)$ consists of the star of a single rose in $S_{s(j)}$, and hence is contractible.

Proof. Suppose that (R, ϕ) is a rose as in Figure 2.9 with

$$l(y_i^j) = 1,\ \forall\, i = 1, \ldots, s(j);\ l(y_i^j y_l^j) = l(y_i^j \overline{y}_l^j) = 2,\ \forall\, i < l,\ i, l = 1, \ldots, s(j). \tag{2.1}$$

Let (R_1, ϕ_1) be another rose in $\mathrm{Minset}(f_j)$. We will prove that the optimal homotopy map f such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "f"] & R_1 \\ R_{s(j)} \arrow[u, "\phi"] \arrow[ur, "\phi_1"'] & \end{tikzcd}$$

is an isometry up to homotopy. First of all, we need to show that $\Gamma_f = R$. Notice that by Proposition 3.15 in [24], the Lipschitz constant is determined by a cycle or a figure eight graph. By (2.1), $\mathrm{Lip}(f) = 1$. By contradiction, if Γ_f is not the whole graph, then there is a loop not in Γ_f that has length less than one by Theorem 19 and that gives a contradiction with our assumption (2.1). Hence, $\Gamma_f = R$. In order to prove that f is an isometry, we need to show that we do not have any illegal turn. If a loop contains an illegal turn, then its length is stretched by a number $< \mathrm{Lip}(f)$ (see [24]). Therefore, if we have a loop with an illegal turn, then the length of that loop would be less than the Lipschitz constant, 1, but that is a contradiction. If $s(j) = 2$, then we have other two possibilities for illegal turns (see Figure 2.10). In both cases the length of the figure eight graph would be less than 2 and that leads to a contradiction. In general, if we have an illegal turn in a path contained in a subrose of m petals, then the sum of the lengths of the edges in the subrose would be less than m, and that contradicts (2.1). In conclusion, f is an isometry and $\mathrm{Minset}(f_j)$ consists of the star of a single rose in $S_{s(j)}$. $\square$

A different proof of that lemma can be found in [22]. Applying Lemma 22 to $\mathrm{Minset}(f_j) \hookrightarrow S_n$, for each (Γ, ψ) in $\mathrm{Minset}(f_j)$, the marking ψ is an embedding on the wedge cycle $\mathbb{B}_j$, and we have the following corollary.

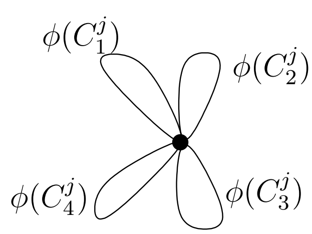

Figure 2.9. The rose (R, ϕ) with $s(j) = 4$.

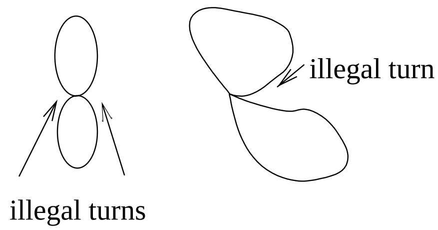

Figure 2.10. Two possible illegal turns.

Corollary 23. $\mathrm{Minset}(f_j) = S_n(A_j)$.

Therefore, by Corollary 23 and Lemma 22, if $k = 1$ (i.e., we have only one wedge cycle), then $S_n(A_1)$ is contractible. Note that $\mathrm{Minset}(f) \subseteq \mathrm{Minset}(f_j)$, for all $j = 1, \ldots, k$. Now it remains to prove that $S_n(\mathcal{A})$ is contractible for $k > 1$. We will follow the approach described in [18].

Definition 24. A forest F in $(\Gamma, \phi) \in \mathrm{Minset}(f)$ is called *admissible* if the marked graph (Γ', ϕ') obtained by collapsing each tree in F to a point is also in $\mathrm{Minset}(f)$.

Lemma 25. Let $(\Gamma, \phi) \in \mathrm{Minset}(f)$ and $\phi(C_i^j)$ be the reduced path representing $\phi(w_i^j)$, for $1 \leq i \leq s(j)$, $1 \leq j \leq k$. Then

- $\mathbb{B}_j = \bigvee_{i=1}^{s(j)} \phi(C_i^j)$ is a wedge cycle in Γ for all $1 \leq j \leq k$;
- $\mathbb{B}_j \cap \mathbb{B}_{j'}$ $(j' \neq j)$ is either empty, a point or a tree;
- $\bigcup(\mathbb{B}_j \cap \mathbb{B}_{j'})$ is a forest in Γ;
- If F is an admissible forest in $\Gamma \setminus \{\phi(C_1^1), \ldots, \phi(C_{s(k)}^k)\}$, then $F \cup \bigcup(\mathbb{B}_j \cap \mathbb{B}_{j'})$ is an admissible forest in Γ.

Proof. Let (R, ψ) be any marked rose in $\mathrm{Minset}(f)$ with (Γ, ϕ) in its star. By Lemma 22, for each (Γ, ϕ) in $\mathrm{Minset}(f)$, the marking ϕ is an embedding on $\bigvee_{i=1}^{s(j)} \phi(C_i^j)$. Thus, $\mathbb{B}_j = \bigvee_{i=1}^{s(j)} \phi(C_i^j)$ is a wedge cycle in Γ for all $1 \leq j \leq k$. Because (Γ, ϕ) is obtained by blowing up the vertex in R into a tree T, the intersection $\mathbb{B}_j \cap \mathbb{B}_{j'}$ is contained in T and it is connected (see Figure 2.11), so the union of all such intersections is a forest in T. The last statement of the lemma follows from the previous observation. □

Hence, because $\mathbb{B}_j \cap \mathbb{B}_{j'}$ can be a tree, $\mathrm{Minset}(f)$ is not contained in $S_n(\mathcal{A})$. However, we have the following theorem.

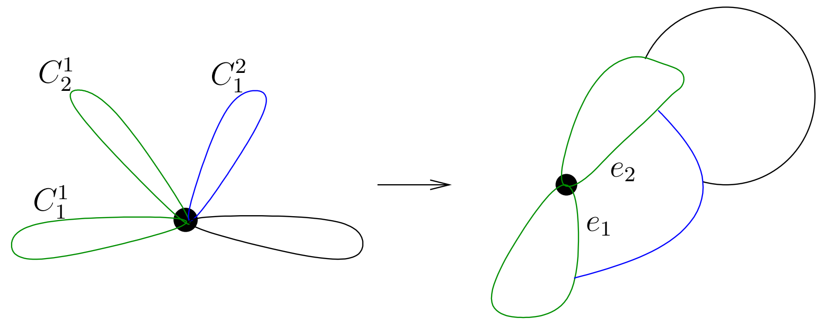

Figure 2.11. A point (Γ, ϕ) in Minset(f), where $A_1 =< y_1^1, y_2^1 >$ and $A_2 =< y_1^2 >$. The union of the edges e_1 and e_2 is the intersection $\mathbb{B}_1 \cap \mathbb{B}_2$.

Theorem 26. Minset(f) deformation retracts onto $S_n(\mathcal{A})$.

Proof. First of all, notice that we have $S_n(\mathcal{A}) \hookrightarrow$ Minset(f). Let $(\Gamma, \phi) \in$ Minset(f). Collapsing each component of $\bigcup(\mathbb{B}_j \cap \mathbb{B}_{j'})$ to a point we obtain a map g from Minset(f) to $S_n(\mathcal{A})$ (see Figure 2.12). If $(\Gamma', \phi') \in$ Minset(f) is obtained from (Γ, ϕ) by collapsing a forest F, then $F \cup \bigcup(\mathbb{B}_j \cap \mathbb{B}_{j'})$ is also a forest in Γ by Lemma 25. Hence, g is a poset map. By the Poset Lemma, g is a deformation retraction from Minset(f) onto $S_n(\mathcal{A})$. $\square$

We are now able to prove Theorem 11.

Proof. By Theorem 26, $S_n(\mathcal{A})$ is a deformation retraction of Minset(f). Because Minset(f) is contractible by Theorem 17, $S_n(\mathcal{A})$ is contractible. We conclude that CV$_n(\mathcal{A})$ is contractible because CV$_n(\mathcal{A})$ deformation retracts onto $S_n(\mathcal{A})$. $\square$

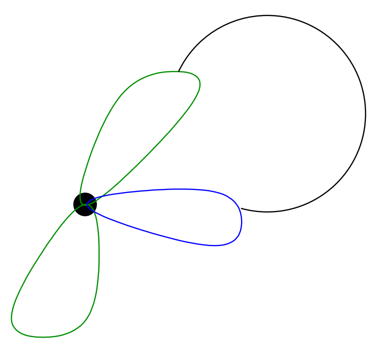

Figure 2.12. The image $g((\Gamma, \phi))$, where $(\Gamma, \phi) \in$ Minset(f) is the point in Figure 2.11, is given by collapsing e_1 and e_2.

2.3 Relative Spine vs. Small Spine

We introduce a new spine, called small spine, that is a simplicial complex smaller than the relative spine, but which carries all the data coming from the relative outer space. Consider the relative outer space $\mathrm{CV}_n(\mathcal{A})$.

Definition 27. Let $D_n(\mathcal{A})$ be the subcomplex of $S_n(\mathcal{A})$ spanned by vertices (Γ, ϕ) in which all the wedge cycles are disjoint. $D_n(\mathcal{A})$ is called *small spine*.

Thus the small spine $D_n(\mathcal{A})$ is a simplicial complex. The definition of small spine shall be more clear after few examples.

Example 28. Suppose $s(i) > 1$ for $i = 1, \dots, k$. Let (Γ, ϕ) be a maximal graph in the centroid of a maximal dimensional open polysimplex in the relative outer space $\mathrm{CV}_n(\mathcal{A})$ with vertices on the wedge cycles. So the basepoints of the wedge cycles have valence $2s(1), \dots, 2s(k)$ and the other vertices have valence 3. The maximal simplex of the small spine that contains Γ is given by the barycentric subdivision of the polysimplex obtained by varying the length of the edges in $\phi(C_1^1), \dots, \phi(C_{s(k)}^k)$ and leaving the length of the edges in $\Gamma' = \Gamma \setminus \{\phi(C_1^1), \dots, \phi(C_{s(k)}^k)\}$ equal to $\frac{1}{N}$, where N is the number of edges in Γ'.

Example 29. Consider the group $\mathrm{Out}(F_4; A_1, A_2)$, where $F_4 =< a, a', b, b' >$ and $A_1 =< a, a' >$, $A_2 =< b, b' >$. Notice that, using Stalling's method (see [41]), an element in $\mathrm{Out}(F_4; A_1, A_2)$ is of the form:

$$\begin{array}{rcl}
a & \mapsto & \omega(a,a')\, a^{\varepsilon_1}\, \overline{\omega}(a,a') \\
a' & \mapsto & \omega(a,a')\, a'^{\varepsilon_2}\, \overline{\omega}(a,a') \\
b & \mapsto & \omega(b,b')\, b^{\varepsilon_3}\, \overline{\omega}(b,b') \\
b' & \mapsto & \omega(b,b')\, b'^{\varepsilon_4}\, \overline{\omega}(b,b')
\end{array}$$

where $\varepsilon_i \in \{\pm 1\}$, $1 \leq i \leq 4$, $\omega(a, a') \in A_1$ and $\omega(b, b') \in A_2$. Modulo separating edges, a point in $\mathrm{CV}_4(A_1, A_2)$ is given by two wedge cycles, with two cycles each, attached in one point (see Figure 2.13). The relative spine and the small spine $D_4(A_1, A_2)$ are both equal to the product of two trees T_1 and T_2 that correspond to the universal coverings of the wedge cycles associated to A_1 and A_2.

Example 30. Consider the group $\mathrm{Out}(F_5; A_1, A_2)$, where $F_5 =< a, a', b, b', c >$ and $A_1 =< a, a' >$, $A_2 =< b, b' >$ (see Example 40). By Example 29 and Stalling's method, an element in $\mathrm{Out}(F_5; A_1, A_2)$ is of the form:

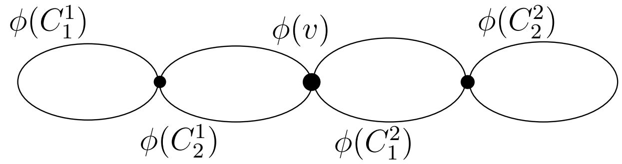

Figure 2.13. A point (Γ, ϕ) in $\mathrm{CV}_4(A_1, A_2)$.

$$
\begin{array}{rcl}
a & \mapsto & \omega(a,a')\, a^{\varepsilon_1}\, \overline{\omega}(a,a') \\
a' & \mapsto & \omega(a,a')\, a'^{\varepsilon_2}\, \overline{\omega}(a,a') \\
b & \mapsto & \omega(b,b')\, b^{\varepsilon_3}\, \overline{\omega}(b,b') \\
b' & \mapsto & \omega(b,b')\, b'^{\varepsilon_4}\, \overline{\omega}(b,b') \\
c & \mapsto & u_1(a,a',b,b') c^{\varepsilon_5} u_2(a,a',b,b')
\end{array}
$$

where $\varepsilon_i \in \{\pm 1\}$, $1 \leq i \leq 5$, $\omega(a,a') \in A_1$, $\omega(b,b') \in A_2$ and $u_i(a,a',b,b')$'s are elements in $F_4 =< a, a', b, b' >$. The relative outer space, the relative spine and the small spine are more complicated than in Example 29, but let us understand what is happening in this case. Consider the point (Γ, ϕ) in $\mathrm{CV}_5(A_1, A_2)$ described in Figure 2.14. Varying the length of the edges in Γ we can move in an open (maximal) 5-polysimplex of $\mathrm{CV}_5(A_1, A_2)$. When we shrink an edge e that is not in a wedge cycle, we end up in an open 4-polysimplex. Because the only way to move away from that open 4-polysimplex is to blow up the vertex given by collapsing e, that open 4-polysimplex is a free face (i.e., it is a face of a unique polysimplex) in the relative outer space. Hence, it is possible to deformation retract the free face onto the interior of the (maximal) polysimplex. First collapsing e and then four edges in the wedge cycles we get a 5-simplex σ in the relative spine. By Definition 27 of small spine, the edge e cannot be collapsed and so σ is not in the small spine. Repeating the same argument for the graph in Figure 2.4 of Example 40, we can compare the simplices in $S_5(A_1, A_2)$ (see Figure 2.8) with the simplices in $D_5(A_1, A_2)$ in Figure 2.15. Therefore, the small spine is strictly smaller than the relative spine, but what are missing are vertices in the relative outer space corresponding to free faces.

Following [18] we will prove that there is a deformation retraction from the relative spine to the small spine.

Theorem 31. There is an $\mathrm{Out}(F_n; \mathcal{A})$-equivariant deformation retraction of $S_n(\mathcal{A})$ onto the small spine $D_n(\mathcal{A})$.

Proof. By the definition of $D_n(\mathcal{A})$, we can build $S_n(\mathcal{A})$ from the small spine $D_n(\mathcal{A})$ by adding marked $(\mathcal{A}, n)$-graphs (Γ, ϕ) in order of decreasing number of vertices in Γ. Thus at

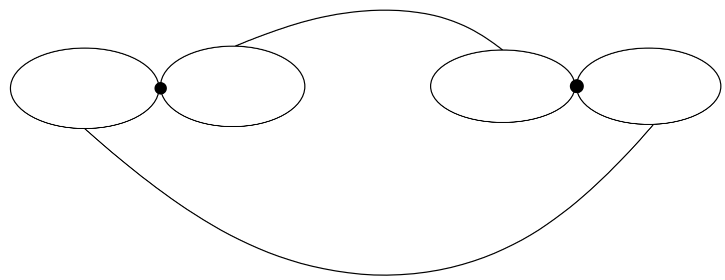

Figure 2.14. A point in $S_5(A_1, A_2)$.

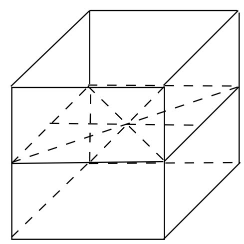

Figure 2.15. The simplices in $D_5(A_1, A_2)$ in this picture are given by the barycentric subdivision of the horizontal square in the middle of the cube.

each stage, we are attaching (Γ, ϕ) along its entire upper link in $S_n(\mathcal{A})$. Hence, it is suffices to show that the upper link is contractible.

Note that a marked $(\mathcal{A}, n)$-graph in $S_n(\mathcal{A})$ with k vertices (the basepoints of the wedge cycles) of valence $2s(1), \ldots, 2s(k)$ and the remaining vertices of valence 3 is in $D_n(\mathcal{A})$. Let $(\Gamma, \phi) \in S_n(\mathcal{A}) \setminus D_n(\mathcal{A})$. Then Γ contains at least one vertex that is in at least two wedge cycles. Let v be one of those vertices. In order to prove that the upper link of (Γ, ϕ) in $S_n(\mathcal{A})$ is contractible, it is suffices to prove the following lemma.

Lemma 32. If v is contained in at least two wedge cycles, then $L(v)$ is contractible.

That lemma can be proved as the Claim in the proof of Proposition 17 in [18]. We will briefly sketch the argument of the proof.

The set of half edges E_v at v is the union of half edges $A = \{a_1, \overline{a}_1, \ldots, a_r, \overline{a}_r\}$ contained in some wedge cycle $\mathbb{B}_i$ and $B = \{b_1, \ldots, b_s\}$ not contained in any wedge cycle. Fix an element $a \in A$ and define the *inside* of an ideal edge to be the side containing a, and the *size* to be the number of half edges on the inside. By hypothesis, $r \geq 2$. The lemma is proved by induction on s. If $s = 0$, consider the ideal edge α that separates a and $\overline{a}$ from all the other half edges. Let $\mathrm{st}(\alpha)$ denote the star of α in $L(v)$. By adding vertices of $L(v) \setminus \mathrm{st}(\alpha)$ in order of increasing size, $L(v)$ deformation retracts onto $\mathrm{st}(\alpha)$. Therefore, $L(v)$ is contractible. The inductive step can be proved in a similar way.

That concludes the proof of the theorem. $\square$

See Figure 2.16 for an example of the deformation retraction of $S_n(\mathcal{A})$ onto $D_n(\mathcal{A})$.

Corollary 33. The small spine $D_n(\mathcal{A})$ is contractible.

2.4 Virtual Cohomological Dimension of $\mathrm{Out}(F_n; \mathcal{A})$

In this section we obtain a corollary of the fact that the relative outer space is contractible computing the virtual cohomological dimension of $\mathrm{Out}(F_n; \mathcal{A})$.

Theorem 34. We have

$$\mathrm{vcd}(\mathrm{Out}(F_n; A_1, \ldots, A_k)) = 2n - 2s(1) - \cdots - 2s(k) + 2k - 2 - m,$$

where $s(i_1) = \cdots = s(i_m) = 1$ and $s(j) > 1$ for $j \neq i_1, \ldots, i_m$.

Proof. Suppose $s(i_1) = \cdots = s(i_m) = 1$ and $s(j) > 1$ for $j \neq i_1, \ldots, i_m$. Recall that we consider $F_n =< y_1^1, \ldots, y_{s(k)}^k, x_1, \ldots, x_{n-\sum_{i=1}^k s(i)} >$.

We denote $\theta^i = (y_1^i, \ldots, y_{s(i)}^i)$. Reordering the y's if necessary, we can suppose $i_1 = k - m + 1, \ldots, i_m = k$. Consider the quotient map $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$. In order to compute the lower bound, we notice that the image of the Abelian subgroup of $\mathrm{Aut}(F_n)$

$$A =< \alpha_i, \beta_i, \gamma_j, \delta_r \,|\, 1 \leq i \leq n - \sum_{i=1}^{k} s(i), 1 < j \leq k, 1 < r \leq k - m >,$$

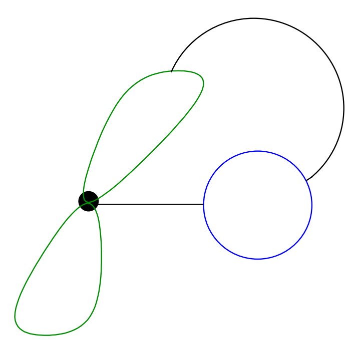

Figure 2.16. The deformation retraction of $S_4(A_1, A_2)$ onto $D_4(A_1, A_2)$ for the graph (Γ, ϕ) described in Figure 2.12.

where α_i fixes all the elements of the basis except $x_i \mapsto y_1^1 x_i$, β_i fixes all the elements of the basis except $x_i \mapsto x_i \bar{y}_1^1$, γ_j fixes all the elements of the basis except $\theta^j \mapsto y_1^1 \theta^j \bar{y}_1^1$ and δ_r fixes all the elements of the basis except $\theta^r \mapsto y_1^r \theta^r \bar{y}_1^r$, is in $\mathrm{Out}(F_n; \mathcal{A})$.

Indeed, obviously $\{\alpha_i\}_{i=1,\ldots,n-\sum_{i=1}^k s(i)}$ and $\{\beta_i\}_{i=1,\ldots,n-\sum_{i=1}^k s(i)}$ commute with all the generators of the subgroup. Because we have

$$\gamma_i \circ \delta_i(y_j^i) = \gamma_i(y_1^i y_j^i \bar{y}_1^i) = y_1^1 y_1^i \bar{y}_1^1 y_1^1 y_j^i \bar{y}_1^1 y_1^1 \bar{y}_1^i \bar{y}_1^1 =$$

$$= y_1^1 y_1^i y_j^i \bar{y}_1^i \bar{y}_1^1 = \delta_i(y_1^1 y_j^i \bar{y}_1^1) = \delta_i \circ \gamma_i(y_j^i),$$

γ_i commutes with δ_j for all i and j. Therefore, A is Abelian. It remains to check that all the basis elements are independent. Notice that $\{\alpha_i, \beta_j\}_{i,j=1,\ldots,n-\sum_{i=1}^k s(i)}$ are independent (the proof is analog to the one for $\mathrm{Out}(F_n)$, see [22]) and that $\bar{y}_1^1 \gamma_j y_1^1$ is the conjugation by $\bar{y}_1^1$ of all the elements except θ^j. Moreover, any compositions of conjugates of elements in the free Abelian subgroup D generated by α_i and β_i for $1 \leq i \leq n - \sum_{i=1}^k s(i)$ cannot be equal to γ_j or δ_r, for all $1 < j \leq k$, $1 < r \leq k - m$. Indeed, for all $1 < j \leq k$, $1 < r \leq k - m$ we have

1. $\gamma_j(x_i) = x_i$, $\delta_r(x_i) = x_i$, $\forall\, i \in \{1, \ldots, n - \sum_{i=1}^k s(i)\}$;
2. $\gamma_j(y_p^1) = y_p^1$, $\delta_r(y_p^1) = y_p^1$, $\forall\, p \in \{1, \ldots, s(1)\}$,

and a composition of conjugates of elements in D contradicts (1) or (2). For example, let F be the conjugate by y_1^1 of

$$\alpha_1^{-1} \circ \cdots \circ \alpha_{n-\sum_{i=1}^k s(i)}^{-1} \circ \beta_1^{-1} \circ \cdots \circ \beta_{n-\sum_{i=1}^k s(i)}^{-1}.$$

We have $F(x_i) = x_i$, but $F(y_p^1) = y_1^1 y_p^1 \bar{y}_1^1$ for $p \in \{1, \ldots, s(1)\}$.

Now we proceed by induction. We start considering the subgroup D_1 generated by γ_2, α_i, β_i for $1 \leq i \leq n - \sum_{i=1}^k s(i)$. An argument similar to the previous one shows that a composition of conjugates of elements in D_1 cannot be equal to γ_3. Hence, $\{\alpha_i, \beta_l, \gamma_2, \gamma_3\}$ are independent. By induction, $\{\alpha_i, \beta_l, \gamma_j\}$ are independent for $1 \leq i, l \leq n - \sum_{i=1}^k s(i)$, $1 < j \leq k$.

Again using a similar argument, it is easy to prove (by induction) that $\{\alpha_i, \beta_l, \gamma_j, \delta_r\}$ are independent for $1 \leq i, l \leq n - \sum_{i=1}^k s(i)$, $1 < j \leq k$, $1 < r \leq k - m$. Hence, there is an Abelian free group of rank $2n - 2\sum_{i=1}^k s(i) + 2k - 2 - m$ contained in our group. For an upper bound, we compute the dimension of the small spine. Suppose that the wedge cycles lie in a maximally blown up graph in the small spine and that the graph has V vertices and E edges. The vertices corresponding to the basepoints of the wedge cycles have valence $2s(1), \ldots, 2s(k-m)$ and the remaining vertices have valence 3. Therefore,

$$E = \frac{3(V-k+m)}{2} + s(1) + \cdots + s(k-m) =$$

$$= \frac{3V - 3k + 3m + 2s(1) + \cdots + 2s(k-m)}{2}.$$

Because $V - E = 1 - n$, we get

$$V = 2n + 3k - 2s(1) - \cdots - 2s(k-m) - 3m - 2.$$

Because the wedge cycles must stay disjoint, we can collapse V vertices to k vertices (which is the number of wedge cycles). Then,

$$\dim(D_n(\mathcal{A})) = 2n + 2k - 2s(1) - \cdots - 2s(k-m) - 3m - 2.$$

Because $s(k-m+1) = \cdots = s(k) = 1$,

$$\begin{aligned} \mathrm{vcd}(\mathrm{Out}(F_n;\mathcal{A})) &\leq 2n + 2k - 2s(1) - \cdots - 2s(k-m) - 3m - 2 = \\ &= 2n - 2\textstyle\sum_{i=1}^{k} s(i) + 2k - 2 - m. \end{aligned}$$

The result follows from Theorem 11. □

Corollary 35. If $m = 0$, then

$$\mathrm{vcd}(\mathrm{Out}(F_n; A_1, \ldots, A_k)) = 2n - 2s(1) - \cdots - 2s(k) + 2k - 2.$$

Our computation of the virtual cohomological dimension of the relative outer space agrees with the computation in [18] when $m = k$. For $k = n$ and $s(1) = \cdots = s(k) = 1$, $\mathrm{Out}(F_n; A_1, \ldots, A_k)$ is called the *pure symmetric automorphism group*. In [21], Collins showed that the virtual cohomological dimension of the pure symmetric automorphism group is $n - 2$.

Remark 36. Suppose that the wedge cycles lie in a maximally blown up graph in c connected components in the relative spine $S_n(\mathcal{A})$ and that the graph has V vertices and E edges. Because two wedge cycles must meet in a valence 4 vertex and the dual graph of the wedge cycles is a forest, there are $k - c$ vertices of valence 4 in the graph. The vertices corresponding to the basepoints of the wedge cycles have valence $2s(1), \dots, 2s(k-m)$. The remaining vertices have valence 3. Therefore,

$$E = \frac{3(V-2k+c+m)}{2} + s(1) + \cdots + s(k-m) + 2(k-c) =$$
$$= \frac{3V-2k-c+3m+2s(1)+\cdots+2s(k-m)}{2}.$$

Because $V - E = 1 - n$, we get

$$V = 2n + 2k + c - 2s(1) - \cdots - 2s(k-m) - 3m - 2.$$

We can collapse V vertices to 1 vertex by first collapsing all except one edge in each cycle $\phi(C_i^j)$ and then collapsing some remaining tree. That gives a simplex of dimension $2n + 2k + c - 2s(1) - \cdots - 2s(k) - 3 - m$ $(c \leq k)$.

Collapse all the separating edges and note that the maximum c is k if $n - \sum_{i=1}^k s(i) \geq 1$ and 1 if $n = \sum_{i=1}^k s(i)$.

If $n - \sum_{i=1}^k s(i) \geq 1$, then the maximum $c = k$ and the dimension of a maximal simplex is $2n + 3k - 2s(1) - \cdots - 2s(k) - 3 - m$. Notice that if $k = 0$, then $m = 0$ and we get the classical result $\mathrm{vcd}(\mathrm{Out}(F_n)) \leq 2n - 3$.

If $n = \sum_{i=1}^k s(i)$, then the maximum $c = 1$ and the dimension of a maximal simplex is $2n + 2k - 2s(1) - \cdots - 2s(k) - 2 - m$. Hence,

$$\dim(S_n(\mathcal{A})) = \begin{cases} 2n + 3k - 2\sum_{i=1}^k s(i) - 3 - m, & \text{if } n - \sum_{i=1}^k s(i) \geq 1 \\ 2n + 2k - 2\sum_{i=1}^k s(i) - 2 - m, & \text{if } n = \sum_{i=1}^k s(i). \end{cases}$$

Notice that if $n = \sum_{i=1}^k s(i)$, then $\dim(S_n(\mathcal{A})) = \dim(D_n(\mathcal{A}))$ (see Example 29).

Because a maximal graph has $3n + 3k - 2\sum_{i=1}^k s(i) - 3 - m$ edges if $n - \sum_{i=1}^k s(i) \geq 1$ and $3n + 2k - 2\sum_{i=1}^k s(i) - 2 - m$ edges if $n = \sum_{i=1}^k s(i)$ and we impose the conditions that the relative volume is 1 (if $n - \sum_{i=1}^k s(i) \geq 1$) and that the sum of the length of the edges in each cycle is 1, we have the following result.

Corollary 37.

$$\dim(\mathrm{CV}_n(\mathcal{A})) = \begin{cases} 3n + 3k - 3\sum_{i=1}^k s(i) - 4 - m, & \text{if } n - \sum_{i=1}^k s(i) \geq 1 \\ 3n + 2k - 3\sum_{i=1}^k s(i) - 2 - m, & \text{if } n = \sum_{i=1}^k s(i). \end{cases}$$

Note that if $k = 0$ and $n > 1$, then $m = 0$ and we have $\dim(\mathrm{CV}_n) = 3n - 4$.

In conclusion, we introduced the contractible relative outer space on which $\mathrm{Out}(F_n; \mathcal{A})$ acts properly discontinuously and with finite stabilizer. We computed the virtual cohomological dimension of $\mathrm{Out}(F_n; \mathcal{A})$, which is a topological invariant of the group, and the dimension of the relative outer space.

CHAPTER 3

GEOMETRY IN THE MODIFIED RELATIVE OUTER SPACE

In this chapter we aim to define a new space on which $\mathrm{Out}(F_n;\mathcal{A})$ acts and analyze its geometry. This new space will be a modified version of the relative outer space that we introduced in the previous chapter. Moreover, we define the train tracks for relative outer automorphisms and the (nonsymmetric) Lipschitz metric on this space generalizing the definition in [13] and some results in [1] and [24] such as the classification of candidates. Then we prove the *Relative Train Track Theorem*: every irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n;\mathcal{A})$ of infinite order has a topological representative which is a train track map. This theorem is the relative version of the analog result in the case of $\mathrm{Out}(F_n)$ (see [4]). However, in the relative case we could not use the fact that the space is locally compact because this is not true for the modified relative outer space. Instead, we prove that there are only a finite number of classes of candidates. Moreover, we study the tangent spaces of the modified relative outer space and we prove that the Lipschitz metric is almost symmetric following the approach in [2]. Finally, we present facts about Nielsen paths, Whitehead graphs and basis elements that we will need in the next chapters.

3.1 Modified Relative Outer Space

We define a new space which is a modified version of the relative outer space as a subset of the compactification of CV_n in which the only elliptic subgroups are $A_i < F_n$, $1 \le i \le k$ (see Proposition 46). We change the definition of a marked metric $(\mathcal{A}, n)$-graph by letting edges in the wedge cycles to have length 0. Basically we think of the wedge cycles as if they are infinitesimally small. The formal definition is the following.

Definition 38. A *modified marked metric* $(\mathcal{A}, n)$*-graph* (Γ, ϕ, l) is a marked graph (with possible separating edges) (Γ, ϕ) such that

- each edge in the wedge cycles has length 0, each edge e in $\widehat{\Gamma}$ (the graph obtained from Γ by collapsing the wedge cycles to special points) has length $\widehat{l}(e) = l_{|\widehat{\Gamma}}(e) \in [0,1]$, and the union of edges in $\widehat{\Gamma}$ with length zero is a forest;

- the wedge cycles are still disjoint after we collapse the edges in $\widehat{\Gamma}$ with length 0.

Definition 39. The *modified relative outer space* $\mathrm{CV}'_n(\mathcal{A})$ is the space of equivalence classes of modified marked metric $(\mathcal{A}, n)$-graphs where

1. the sum of all lengths of the edges in $\widehat{\Gamma}$ is 1 (relative volume 1);
2. $(\Gamma_1, \phi_1) \sim (\Gamma_2, \phi_2)$ if there is a map $h : \Gamma_1 \to \Gamma_2$ such that if Γ'_1 and Γ'_2 are the graphs obtained by collapsing the edges and the preimages of the edges of length 0 in Γ_1 and Γ_2, respectively, then h induces an isometry $h' : \Gamma'_1 \to \Gamma'_2$, $h \circ \phi_1(C_i^j) = \phi_2(C_i^j)$, $\forall\, i, j$ and $h \circ \phi_1$ is homotopic to ϕ_2 rel. C_i^j, $\forall\, i, j$.

Notation 1. We will usually denote a point in $\mathrm{CV}'_n(\mathcal{A})$ by $X = (\Gamma, \phi)$. If we want to stress on the length of the edges in the graph Γ, then we write a point in $\mathrm{CV}'_n(\mathcal{A})$ as (Γ, ϕ, l).

Let $R'_n(\mathcal{A})$ be the rose $R_n(\mathcal{A})$ with each edge in the wedge cycles of length 0. There is a natural right action of $\mathrm{Out}(F_n; \mathcal{A})$ on $\mathrm{CV}'_n(\mathcal{A})$ given by changing the marking: let $X = (\Gamma, \phi) \in \mathrm{CV}'_n(\mathcal{A})$ and $\Psi \in \mathrm{Out}(F_n; \mathcal{A})$, consider $\psi : R'_n(\mathcal{A}) \to R'_n(\mathcal{A})$ such that $[\psi_*] = \Psi$. The right action is given by

$$X \cdot \Psi = (\Gamma, \phi) \cdot \Psi = (\Gamma, \phi \circ \psi).$$

However, notice that the stabilizer of a point is infinite (see Example 40).

We define a topology on $\mathrm{CV}'_n(\mathcal{A})$ by varying the length of the edges that are not in any wedge cycle. Since the sum of the lengths of these edges is 1, $\mathrm{CV}'_n(\mathcal{A})$ is a simplicial complex with missing faces. In other words, we can think of the modified relative outer space as the union of open simplices.

We define the *modified relative spine* $S'_n(\mathcal{A})$ of the modified relative outer space as the geometric realization of the partially ordered set of open simplices. Notice that $S'_n(\mathcal{A})$ is a simplicial complex.

Example 40. Consider $\mathrm{Out}(F_2; A)$, where $F_2 =< a, b >$, $A =< a >$. The modified relative outer space $\mathrm{CV}'_2(A)$ is a point X with an infinite countable number of half-open edges attached. The modified relative outer space $\mathrm{CV}'_2(A)$ can be viewed as a subset of the compactification of CV_2. The action of the group on $\mathrm{CV}'_2(A)$ is given by rotating the edges. Hence, the stabilizer of X is $\mathrm{Out}(F_2; A)$. Moreover, $\mathrm{CV}'_2(A)/\mathrm{Out}(F_2; A)$ is a 1-simplex with a missing vertex. In Example 9 we have seen that the relative spine $S_2(A)$ is naturally homeomorphic to a line. The modified relative spine $S'_2(A)$ is a point with an infinite number of closed 1-simplices coming out from that point (see Figure 3.1).

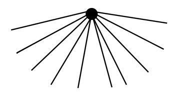

Figure 3.1. The modified relative spine $S_2'(A)$.

Example 41. Consider $\mathrm{Out}(F_n;\mathcal{A})$, where $n-\sum_{i=1}^{k} s(i)=1$. A graph in the modified relative spine $S_n'(\mathcal{A})$, modulo separating edges, is a loop with i vertices, $1\le i\le k$, that correspond to the wedge cycles (see Figure 3.2).

Note that if $B=1$ and $k=1$ or $k=2$, $\mathrm{CV}_n'(\mathcal{A})$ is a single point. An unprojectivized definition of the modified relative outer space is the following.

Definition 42. A *homothety* between two metric graphs Γ_1 and Γ_2 is a homeomorphism $h:\Gamma_1\to\Gamma_2$ such that

$$d_{\Gamma_2}(h(x),h(y))=c\cdot d_{\Gamma_1}(x,y)$$

for all $x,y\in\Gamma_1$ and for some constant $c>0$ called *stretch factor* of h.

The unprojectivized modified relative outer space is the space of equivalence classes of modified marked metric $(\mathcal{A},n)$-graphs with relative volume 1 and the following equivalence relation: $(\Gamma_1,\phi_1)\simeq(\Gamma_2,\phi_2)$ if there is a homothety $h':\Gamma_1'\to\Gamma_2'$ with $h\circ\phi_1$ homotopic to ϕ_2 rel. C_i^j, $\forall\, i,j$.

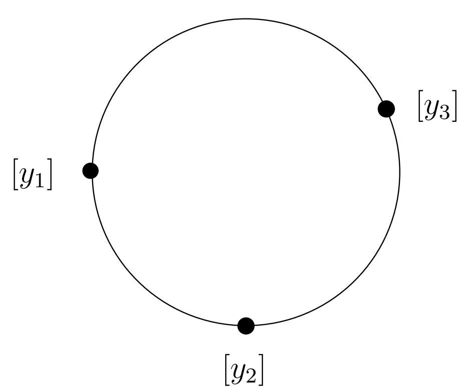

Figure 3.2. A graph in the modified relative spine $S_4'(A_1,A_2,A_3)$, where $A_i=<y_i>$, $1\le i\le 3$.

3.1.1 Contractibility of the Modified Relative Outer Space

The goal of this section is to prove the following theorem.

Theorem 43. The modified relative outer space $\mathrm{CV}'_n(\mathcal{A})$ is contractible.

In order to prove Theorem 43 we need to introduce a new space homeomorphic to the modified relative outer space and prove that this new space is contractible. First, we recall the main definitions and some results for actions on $\mathbb{R}$-trees.

Definition 44. Let (X, d) be a metric space. We say that (X, d) is an $\mathbb{R}$-*tree* if for any $x, y \in X$ there is a unique arc from x to y and this arc is a geodesic segment.

Let $\phi : T \to T$ be an isometry of an $\mathbb{R}$-tree T. The *translation length* of ϕ is

$$l(\phi) = \inf\{d(x, \phi(x)) \,|\, x \in T\}.$$

The infimum is always attained. If $l(\phi) > 0$, there is a unique ϕ-invariant line called the axis of ϕ, and $\phi_{|\text{axis}}$ is a translation by $l(\phi)$. In this case, we say that ϕ is *hyperbolic*. If $l(\phi) = 0$, then ϕ fixes a nonempty subtree of T and is said to be *elliptic*. Let G be a group acting by isometries on an $\mathbb{R}$-tree T. A tree equipped with an isometric action is called *G-tree*. The action is *nontrivial* if no point of T are fixed by the whole group. It is *minimal* if there is no proper G-invariant subtree. The action is *free* if any nonidentity group element does not leave an element of T fixed. Let $Gx = \{gx \,|\, g \in G\}$ be the orbit of $x \in T$. An action of G on T has *dense orbits* if the closure of Gx is the whole tree T. The notion of a deformation space was introduced by Forester [23]. By definition, two G-trees are in the same deformation space if they have the same elliptic subgroups, i.e., if a subgroup of G fixes one point in a tree, it also fixes the image of that point in any other tree. Identifying two trees if they differ only by rescaling the metric leads to the projectivized deformation space. We endow this space with the weak topology. Guirardel and Levitt [27] and Clay [19] proved the contractibility of this space. We will prove that $\mathrm{CV}'_n(\mathcal{A})$ is a projectivized deformation space.

Let the *deformation space* $\mathcal{D}$ be the space of simplicial F_n-trees with elliptic subgroups $A_1, \ldots, A_k$. Let $(\Gamma, \phi) \in \mathrm{CV}'_n(\mathcal{A})$. The tree T_1 associated to (Γ, ϕ) is constructed in the following way. Let Γ_0 be the graph obtained by Γ changing the length of the wedge cycles from 0 to a constant $\varepsilon > 0$. Consider the universal cover $\widetilde{\Gamma_0}$ of Γ_0 and collapse all the rays that correspond to words $a_{i_1} a_{i_2} a_{i_3} \cdots$, $a_{i_j} \in \mathcal{A}$ and its translates. We will call T_1 a tree with special vertices.

Example 45. Consider $\mathrm{Out}(F_2; A)$, where $F_2 =< a, b >$, $A =< a >$, and consider the point $(\Gamma, \phi = \mathrm{id}) \in \mathrm{CV}'_2(A)$ consisting of a loop corresponding to b and a vertex corresponding to a. The graph Γ_0 is the graph in Figure 3.3.

In order to construct the tree T_1 associated to this graph, first we consider its universal covering (i.e., the standard tree T for outer space, see Figure 3.4) and then we collapse the a-axis and its translated axes. Notice that there are infinitely many edges coming out from each vertex (see Figure 3.5).

Notation 2. We will denote $A_1 * \cdots * A_k * B$ by $\mathcal{A} * B$.

Proposition 46. $\mathrm{CV}'_n(\mathcal{A})$ is homeomorphic to $\mathcal{D}$.

Proof. Let $F : \mathrm{CV}'_n(\mathcal{A}) \to \mathcal{D}$ be the map that sends a point $(\Gamma, \phi, l) \in \mathrm{CV}'_n(\mathcal{A})$ to the metric $(\mathcal{A} * B)$-tree T_1 defined previously. It is easy to check that $T_1 \in \mathcal{D}$. The metric on

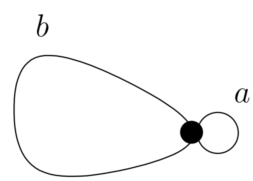

Figure 3.3. The graph Γ_0 in Example 45.

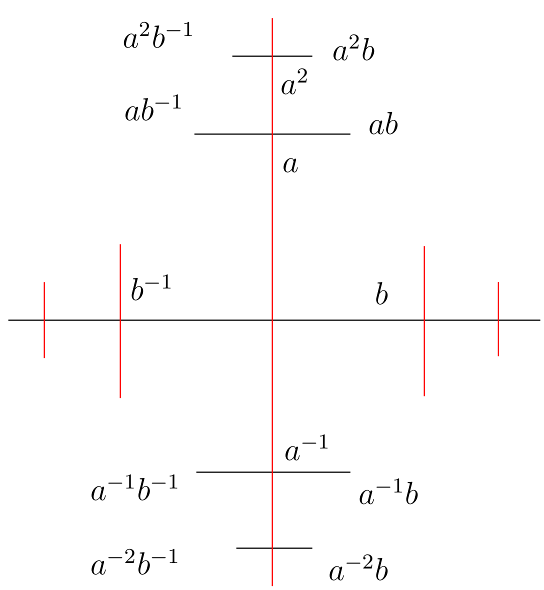

Figure 3.4. The tree T universal covering of Γ_0 in Example 45.

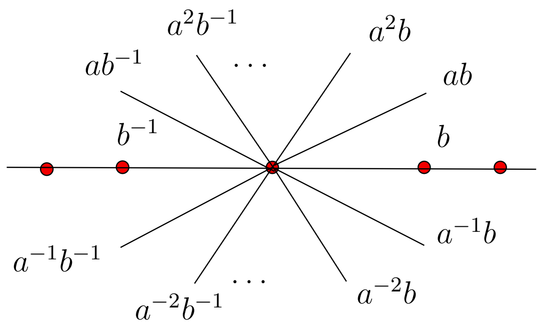

Figure 3.5. The tree T_1 associated to (Γ, ϕ) in Example 45.

T_1 is induced by l and the action on the tree is induced by ϕ.

Obviously, F is a continuous map (see [40]) and its inverse $F^{-1} : \mathcal{D} \to \mathrm{CV}'_n(\mathcal{A})$ is defined in the following way. Let $T \in \mathcal{D}$. Define $F^{-1}(T) = T/(\mathcal{A} * B)$. The quotient $T/(\mathcal{A} * B)$ is a marking metric graph with the metric induced by the metric on T, and the marking determined by the action of $(\mathcal{A} * B)$ on T (see Section 4 in [27]). Moreover, F^{-1} is a continuous map. Therefore, F is a homeomorphism. $\square$

Lemma 47. The deformation space $\mathcal{D}$ is contractible.

See Theorem 6.1 in [27] for a proof of Lemma 47. Theorem 43 follows from Proposition 46 and Lemma 47.

Remark 48. Note that the modified relative outer space is contractible, but not locally compact. For example, if $n = 2$, $k = 1$, and $s(1) = 1$, the space is a point with an infinite number of segments attached. Hence, the space is not locally compact.

Note that by the definition of modified relative outer space, if $\mathcal{A} \neq 1$ we can inject $\mathrm{CV}'_n(\mathcal{A}) \hookrightarrow \partial \mathrm{CV}_n$. In particular, the image of the embedding is contained in the set of points of $\partial \mathrm{CV}_n$ such that $A_1, \dots, A_k$ are the only elliptic subgroups. See[25], [26], and [6] for a description of $\partial \mathrm{CV}_n$. However, $\mathrm{CV}'_n(\mathcal{A})$ endowed with the simplicial topology is not homeomorphic to $\mathrm{CV}'_n(\mathcal{A})$ endowed with the length function topology (see Section 4.2.1).

3.1.2 Dimension of the Modified Relative Outer Space

We compute the dimension of the modified relative spine and the dimension of the modified relative outer space. First, we can deformation retract the spine onto the subcomplex L with vertices (Γ, ϕ), where $\widehat{\Gamma}$ has no edges of length 0. Suppose that Γ is a maximal graph in L (i.e., it has the maximum number of vertices) and consider $\widehat{\Gamma}$. Denote by V and E the number of vertices and edges of $\widehat{\Gamma}$ respectively. The vertices corresponding to the special points have valence 1 and the remaining vertices have valence 3. Hence,

$$E = \frac{3(V-k)}{2} + \frac{k}{2}.$$

Because $V - E = 1 - (n - \sum_{i=1}^{k} s(i))$, we have

$$V = 2n + 2k - 2 - 2\sum_{i=1}^{k} s(i)$$

and hence (because we can collapse V vertices to $s = \max\{k, 1\}$ vertices),

$$\dim(S'_n(\mathcal{A})) = 2n + 2k - 2 - 2\sum_{i=1}^{k} s(i) - s.$$

Notice that if $k = 0$, $\dim(S'_n(1)) = 2n - 3 = \dim(S_n)$. Moreover,

$$E = 3n + 2k - 3 - 3\sum_{i=1}^{k} s(i)$$

and because the relative volume of each graph is 1,

$$\dim(\mathrm{CV}'_n(\mathcal{A})) = 3n + 2k - 4 - 3\sum_{i=1}^{k} s(i).$$

Notice that if $k = 0$, $\dim(\mathrm{CV}'_n(1)) = 3n - 4 = \dim(\mathrm{CV}_n)$. Indeed, if $k = 0$ the modified relative outer space is the standard outer space.

3.1.3 Kernel of the Action

We will determine the kernel KA of the action and we will give an easy description of it. We start with the following example.

Example 49. Consider $F_2 = A * B$, where $A =< a >$ and $B =< b >$. As we noticed in Example 9, $\mathrm{Out}(F_2; A)$ is isomorphic to the infinite dihedral group D_∞ and we can consider the standard representative of a class $[f] \in \mathrm{Out}(F_2; A)$ given by $f_N(a) = a$, and $f_N(b) = a^N b$ or $f_N(b) = a^N b^{-1}$.

Recall that $\mathrm{CV}'_2(A)$ is a point with an infinite number of half-open 1-simplices attached (see Example 40). The action of $\mathrm{Out}(F_2; A)$ on $(\Gamma, f_r, l) \in \mathrm{CV}'_2(A)$ is the following:

$$(\Gamma, f_r, l) \cdot [f_m] = (\Gamma, f_{r+m}, l).$$

Therefore, $[f_m] \in \mathrm{KA}$ if and only if $m = 0$. In this case the kernel of the action is trivial. The action is not proper since we have infinite point stabilizers and in general also the action modulo KA is not proper.

Consider $F_n = \mathcal{A} * B =< y_1^1, \ldots, y_{s(k)}^k, x_1, \ldots, x_{n-\sum_{i=1}^k s(i)} >$. We denote by θ^i the point $(y_1^i, \ldots, y_{s(i)}^i)$. Let $s(i_1) = \cdots = s(i_m) = 1$ and $s(j) > 1$ for $j \neq i_1, \ldots, i_m$.

Proposition 50. 1. If $B = 1$, $k = 1$ or $k = 2$, then $\mathrm{CV}'_n(\mathcal{A})$ is a point and KA is exactly the group $\mathrm{Out}(F_n; \mathcal{A})$.

2. If $B \neq 1$ or $B = 1$ and $n \geq 3$, the kernel of the action is generated by the maps $[f_{i,\alpha}]$, $i \neq i_1, \ldots, i_m$ and $1 \leq \alpha \leq s(i)$, such that $f_{i,\alpha} : \theta^i \mapsto y_\alpha^i \theta^i \overline{y}_\alpha^i$ and $f_{i,\alpha}$ is the identity on the other generators.

Proof. 1. It is obvious by the definition of $\mathrm{CV}'_n(\mathcal{A})$.

2. Suppose that $B \neq 1$. If $k = 0$, then $\mathrm{CV}'_n(\mathcal{A}) = \mathrm{CV}_n$ and hence $\mathrm{KA} = \{1\}$. Let $k \geq 1$. First note that $[f_{i,\alpha}] \in \mathrm{KA}$. Indeed, since the length of the edges in the wedge cycle is 0, $X \cdot [f_{i,\alpha}] = X$, for any $X \in \mathrm{CV}'_n(\mathcal{A})$. Hence, KA contains the subgroup M generated by $\{[f_{i,\alpha}]\}$ and it is easy to see that this is a normal subgroup. We need to show that any $[f] \notin M$ is not in the kernel. Let $[f] \notin M$. We normalize $[f] \in \mathrm{Out}(F_n; \mathcal{A})$ by sending $y_1^1 \mapsto y_1^1$. Let f denote the normalized representative. First note that if f is not the identity on B, then there exist two distinct points X_1 and X_2 in $\mathrm{CV}'_n(\mathcal{A})$ such that $X_1 \cdot [f] = X_2$. For example, let X_1 be the relative rose with each (positive) length $= \frac{1}{n}$ and the identity marking, and let X_2 be the relative rose with the same metric but f as the marking (see also Example 40). Now suppose that f is the identity on B. Let $y_p^j \in A_j$ so that $f : y_p^j \mapsto \omega y_p^j \overline{\omega}$. We can find a graph and a loop α such that α and $f(\alpha)$ do not have the same length. We have two cases:

- $\omega = x_r^\varepsilon \cdots$ ($\varepsilon \in \{\pm 1\}$) and without loss of generality we can suppose $\varepsilon = 1$. Consider the point $X \in \mathrm{CV}'_n(\mathcal{A})$ in Figure 3.6 and the loop $\alpha = y_p^j y_1^1 x_r$. If $l(\alpha) = L$, then $l(f(\alpha)) \geq 3L$. Hence, $[f] \notin \mathrm{KA}$.

- $\omega = (y_j^r)^\varepsilon \cdots$ ($\varepsilon \in \{\pm 1\}$) and without loss of generality we can suppose $\varepsilon = 1$ and $r \neq 1$, $j \neq 1$. Consider the relative rose with each edge of (positive) length

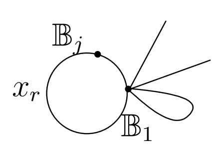

Figure 3.6. The point $X = (\Gamma, \phi)$, where ϕ is the identity marking and the length of the edges is $\frac{1}{n}$.

$= \frac{1}{n}$, and the identity marking. The loop $\alpha = y_p^j y_1^1$ has length $\frac{4}{n}$, while the length of $f(\alpha) = \omega y_p^j \overline{\omega} y_1^1$ as length at least $\frac{5}{n}$. Therefore, $[f] \notin \mathrm{KA}$.

Because the argument for the case $B = 1$ and $k \geq 3$ is similar, we omit the proof.

$\square$

Corollary 51. If $B \neq 1$ or $B = 1$ and $n \geq 3$, the group KA is isomorphic to the product $A_1 \times \cdots \times A_k$.

Proof. By Proposition 50, KA is generated by the maps $[f_{i,\alpha}]$, $i \neq i_1, \ldots, i_m$ and $1 \leq \alpha \leq s(i)$, such that $f_{i,\alpha} : \theta^i \mapsto y_\alpha^i \theta^i \overline{y}_\alpha^i$ and $f_{i,\alpha}$ is the identity on the other generators. Define

$$\begin{array}{rccc} \varphi : & \mathrm{KA} & \rightarrow & A_1 \times \cdots \times A_i \times \cdots \times A_k \\ & f_{i,\alpha} & \mapsto & (0, \ldots, y_\alpha^i, \ldots, 0) \end{array}$$

The map φ is a well-defined homomorphism. The inverse map is

$$\begin{array}{rccc} \varphi^{-1} : & A_1 \times \cdots \times A_k & \rightarrow & \mathrm{KA} \\ & (y_{\alpha_1}^1, \ldots, y_{\alpha_k}^k) & \mapsto & f_{1,\alpha_1} \circ \cdots \circ f_{k,\alpha_k} \end{array}$$

Hence, $\mathrm{KA} \cong A_1 \times \cdots \times A_k$. $\square$

3.2 Train Tracks for Relative Outer Automorphisms

Let (Γ, ϕ) be a modified marked $(\mathcal{A}, n)$-graph. A homotopy equivalence $\psi : \Gamma \rightarrow \Gamma$ that fixes the wedge cycles determines a relative outer automorphism $\Psi \in \mathrm{Out}(F_n; \mathcal{A})$. We suppose that $\widehat{\Gamma}$ does not have edges of length 0. Otherwise, we collapse those edges to points. We extend definitions and facts in [13] to the relative case.

Definition 52. If $\psi : \Gamma \rightarrow \Gamma$ is a homotopy equivalence such that $\psi(v)$ is a vertex for all the vertices $v \in \Gamma$, ψ restricted to $\Gamma \setminus \{$ vertices $\}$ is locally injective and $\psi(\phi(C_i^j)) = \phi(C_i^j)$ for all $1 \leq i \leq s(j)$, $1 \leq j \leq k$, then we say that ψ is a *topological representative* of the relative outer automorphism Ψ.

Enumerate the edges of Γ outside the wedge cycles. A *relative transition matrix* M associated to $\psi : \Gamma \rightarrow \Gamma$ has entries a_{ij} defined as the number of times that the ψ-image of the jth edge crosses the ith edge in either direction and both the edges are not in any wedge cycle.

Example 53. Consider $(\Gamma, \phi) \in \mathrm{CV}_{3,1}(A_1)$, where $A_1 =< y_1 >$, as in Figure 3.7, where $\phi_*(y_1) = a$, $\phi_*(x_1) = b$, and $\phi_*(x_2) = c$ and ψ is defined in the following way:

$$\psi(a) = a,\ \psi(b) = bac,\ \psi(c) = cbac.$$

The relative transition matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

A nonnegative integral matrix M is *irreducible* if for all $1 \leq i, j \leq \dim(M)$, there exists $N(i,j) > 0$ so that the ijth entry of $M^{N(i,j)}$ is positive (see [39]). A proper subgraph of a graph Γ is *nontrivial* if at least one of its components is not a vertex. A subgraph Γ_0 is *invariant* with respect to a topological representative $\psi : \Gamma \to \Gamma$ if $\psi(\Gamma_0) \subset \Gamma_0$. A topological representative $\psi : \Gamma \to \Gamma$ is *irreducible* if the only ψ-invariant nontrivial subgraphs of Γ are contained in the wedge cycles. Equivalently, $\psi : \Gamma \to \Gamma$ is irreducible if and only if its relative transition matrix is irreducible. For example, the topological representative ψ in Example 53 is irreducible. Let (Γ, ϕ) be a marked $(\mathcal{A}, n)$-graph. Consider the graph $\widehat{\Gamma}$ given by collapsing the wedge cycles to points in Γ, denote by $\mathrm{col} : \Gamma \to \widehat{\Gamma}$ the collapsing map and let $\widehat{\phi} = \mathrm{col} \circ \phi$. A forest F in $\widehat{\Gamma}$ is called *essential* if at least two special points are contained in one connected component of F. Otherwise, F is called *nonessential*.

Definition 54. A relative outer automorphism $\Psi \in \mathrm{Out}(F_n; \mathcal{A})$ is *irreducible* if every topological representative $\psi : \Gamma \to \Gamma$ that has no invariant nontrivial nonessential forest in $\widehat{\Gamma}$ is irreducible. Otherwise, Ψ is called *reducible*.

Example 55. Consider $F_3 =< a_1 > * < a_2 > * < a_3 >$ and $B = 1$. Let Γ be the graph consisting of a tripod with edges e_1, e_2, and e_3 of length $\frac{1}{3}$ and a cycle C_i corresponding to a_i at the extremity of the edge e_i $(i = 1, 2, 3)$. Let $f : \Gamma \to \Gamma$ be the following map

$$f : \begin{cases} e_1 \mapsto e_1 \\ e_2 \mapsto e_3 C_2 \overline{e_3} e_2 \\ e_3 \mapsto e_3 \end{cases}$$

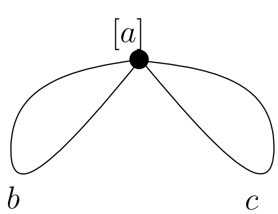

Figure 3.7. The graph (Γ, ϕ), where $\phi_*(y_1) = a$, $\phi_*(x_1) = b$ and $\phi_*(x_2) = c$.

The map f is the train track map for the reducible relative outer automorphism

$$\Phi : \begin{cases} a_1 \mapsto a_1 \\ a_2 \mapsto a_3 a_2 \overline{a_3} \\ a_3 \mapsto a_3 \end{cases}$$

It is reducible because the subgraph $\Gamma_0 = e_1 \cup e_3$ is a nontrivial invariant (with respect to f) essential tree.

Definition 56. A relative outer automorphism $\Psi \in \mathrm{Out}(F_n; \mathcal{A})$ is *fully irreducible*, also called IWIP (irreducible with irreducible powers), if for all $i > 0$, Ψ^i is irreducible.

Remark 57. The relative outer automorphism $\Psi \in \mathrm{Out}(F_n; \mathcal{A})$ is reducible if there is $(\Gamma, \phi) \in S'_n(\mathcal{A})$ and a topological representative $\psi : \Gamma \to \Gamma$ of Ψ such that (Γ, ϕ) contains a nontrivial invariant subgraph Γ_0 not included in the wedge cycles, but does not contain a nontrivial invariant nonessential forest in $\widehat{\Gamma}$. We call such ψ a *reduction* for Ψ.

Remark 58. If there is a proper free factor F_m of B that is invariant up to conjugacy, under the action of Ψ, then Ψ is reducible. Indeed, choose a relative automorphism $\phi : F_n \to F_n$ that represents Ψ such that $\phi(F_m) = F_m$. Choose a free factor F such that

$$F_n \cong \mathcal{A} * F_m * F.$$

Let R_j be a rose with j petals. Identify $\pi_1(R_{n-\sum_{i=1}^k s(i)}, v)$ with B so that the first m edges of the rose correspond to F_m and denote the rose with these edges R_m, and the remaining $n - \sum_{i=1}^k s(i) - m$ edges correspond to F. Then $\phi : F_n \to F_n$ is represented by a homotopy equivalence preserving the wedge cycles $\psi : R_n \to R_n$ that has R_m has a nontrivial invariant subgraph in $\widehat{R_n}$. Hence, Ψ is reducible.

However, it is not true that all the reducible relative outer automorphisms are obtained by fixing a proper free factor F_m of B.

Example 59. Consider a relative rose $(\Gamma, \phi) \in \mathrm{CV}_{4,1}(A_1)$, where $A_1 =< a_1, a_2 >$ and $B =< b_1, b_2 >$ and let $\psi : \Gamma \to \Gamma$ be

$$\psi(a_1) = a_1,\ \psi(a_2) = a_2,\ \psi(b_1) = b_1 a_1,\ \psi(b_2) = b_2 a_2.$$

In this case, ψ is reducible but the ψ-invariant subgraphs $\{a_1, b_1\}$, $\{a_2, b_2\}$ are not contained in wedge cycles.

Example 60. Consider a rose $(\Gamma, \phi) \in \mathrm{CV}_{2,1}(A_1)$, where $A_1 =< a >$ and $B =< b >$ and $\psi : \Gamma \to \Gamma$ such that

$$\psi(a) = a,\ \psi(b) = ab.$$

In this case, ψ is irreducible since the only ψ-invariant subgraph is the cycle corresponding to a.

To each irreducible topological representative $\psi : \Gamma \to \Gamma$ of a irreducible relative outer automorphism, we may assign a Perron-Frobenius eigenvalue λ for the relative transition matrix of $\psi : \Gamma \to \Gamma$. The eigenvalue λ is also called *expansion factor*. As for fully irreducible $\Phi \in \mathrm{Out}(F_n)$, if Φ is a fully irreducible relative outer automorphism, the expansion factor λ of Φ and the expansion factor μ of Φ^{-1} are in general different (see [1] and [28]). A *turn* in (Γ, ϕ) is an unordered pair of oriented edges of $\widehat{\Gamma}$ originating at a common vertex. A turn is *nondegenerate* if it is defined by distinct oriented edges. Otherwise, the turn is called *degenerate*. A map $\psi : \Gamma \to \Gamma$ induces a map $D\widehat{\psi}$ from the set of oriented edges of $\widehat{\Gamma}$ to itself by sending an oriented edge to the first oriented edge in its $\widehat{\psi}$-image. We can think of $D\widehat{\psi}$ as a sort of derivative. $D\widehat{\psi}$ induces a map $T\widehat{\psi}$ on the set of turns in $\widehat{\Gamma}$. A turn at a vertex that is not a special point is *illegal* with respect to $\widehat{\psi}$ if its image under some iterate of $T\widehat{\psi}$ is degenerate. A turn is *legal* if it is not illegal. We say that at the special point $[\mathbb{B}_j]$, if e is an edge in $\widehat{\Gamma}$ the turn $e\alpha\overline{e}$ is legal if and only if α is not trivial in $\mathbb{B}_j$.

A *path* $\alpha : [0,1] \to \Gamma$ is a map that is constant or locally injective. If it is a constant map, then we say that α is a *trivial path*. If α is a nontrivial path in Γ, then $\bigcup\{\alpha^{-1}(v) \,|\, v \text{ is a vertex }\}$ subdivides α into a concatenation of subpaths

$$\alpha = \alpha_1 \cdot \alpha_2 \cdots \alpha_m,$$

where for $2 \leq i \leq m-1$, each α_i maps onto a single edge of Γ, α_1 is the terminal segment contained in some edge, and α_m is the initial segment contained in some edge. The sequence $\alpha_1 \cdots \alpha_m$ is called *edge path* of α, and m is the *combinatorial length* of α and it is denoted by $\mathrm{combl}(\alpha)$.

Let $\widehat{\alpha}$ be the path given by α suppressing the α_i's in the wedge cycles and denoting them as special points. We will not distinguish between $\widehat{\alpha}$ and the concatenation of the edges in its image. Therefore, we say that $\widehat{\alpha}$ contains the turn $\{\overline{e}_i, e_{i+1}\}$, where e_i and e_{i+1} are consecutive edges in $\widehat{\alpha}$. Every map $\sigma : [0,1] \to \Gamma$ is homotopic relatively to the endpoints and to the wedge cycles to a (possibly trivial) path $[\sigma]$. We say that $[\sigma]$ is obtained from σ by tightening. A path α is *legal* if $\widehat{\alpha}$ does not contain any illegal turn. We say that $\alpha : S^1 \to \Gamma$ is a *loop* if it is locally injective. A loop α is *legal* in Γ if α does not contain any illegal turn. Every map $\sigma : S^1 \to \Gamma$ is homotopic to a (possibly trivial) loop $[\sigma]$. We say that $[\sigma]$ is obtained from σ by tightening. A loop α is *legal* in Γ if $\widehat{\alpha}$ does not contain any illegal turn. A *core (sub)graph* is a (sub)graph Δ with all vertices (not special points) of valence ≥ 2. If the $\widehat{\psi}$-image of each edge in each core subgraph $\Delta \subset \widehat{\Gamma}$ is a legal path in Γ, then we say that $\psi : \Gamma \to \Gamma$ is a *train track map*. An equivalent definition is the following. A *direction* at $x \in \widehat{\Gamma}$ is a germ of isometric embeddings $d : [0, \varepsilon] \to \widehat{\Gamma}$ with $d(0) = x$. Most points of $\widehat{\Gamma}$ are not vertices and hence they have only two directions. The number of directions at a vertex v is the valence of the vertex. Directions can be viewed as analog of unit tangent vectors. If $\phi : \Gamma \to \Gamma'$ is a map linear on edges and $\widehat{\phi}(x) = x'$, then ϕ induces a map $\widehat{\phi}_*$ from the set of directions at x to the set of directions at x'. A *train track structure* on a core graph Δ is an equivalence relation on the set of directions at every vertex $v \in \Delta$ with at least two equivalence classes at every vertex. The equivalence classes of edges are called *gates*. If v is a special point $[\mathbb{B}_j]$, $\alpha \in \mathbb{B}_j$ and there is only a direction d at $[\mathbb{B}_j]$, then $d \nsim d$ if ϕ sends the edge e corresponding to the direction d to $e\alpha\overline{e}$. A turn $\{d_1, d_2\}$ is *illegal* if $d_1 \sim d_2$. Otherwise, the turn is *legal*. An immersed loop or a path is legal if it takes only legal turns. The map ψ is a train track map if it preserves the train track structure. We will prove that each irreducible relative outer automorphism of infinite order has a topological representative which is a train track map (see Theorem 66). Let $f : \Gamma \to \Gamma$ be a train track map for an irreducible relative outer automorphism Φ and let λ be the expansion factor. As described in [8] and [9], if $\gamma = \alpha \cdot \beta$ is a path in Γ and α and β are legal, then there exists a constant called bounded cancelation constant $\mathrm{BCC}(f)$ such that $[f(\gamma)]$ is obtained from $f(\alpha) \cdot f(\beta)$ canceling a path given by the terminal end of $f(\alpha)$ and the initial end of $f(\beta)$ of length $\leq \mathrm{BCC}(f)$. Now consider a path $\delta = \alpha \cdot \beta \cdot \gamma$, where α, β, γ are legal paths with respect to f but the turn denoted by the dot might be illegal. Notice that $[f(\delta)]$ contains the subpath of $f(\beta)$ obtained from $f(\beta)$ by truncating paths of length $\mathrm{BCC}(f)$ at both ends and denoted by θ. If $l(\beta) > \frac{2\mathrm{BCC}(f)}{\lambda - 1}$, then $\alpha \cdot \beta \cdot \gamma$ will produce paths with the length of the legal segment corresponding to β going to infinity under iteration. We call the constant $\frac{2\mathrm{BCC}(f)}{\lambda - 1}$ the *critical constant*.

If $l(\beta) > \frac{4\mathrm{BCC}(f)}{\lambda - 1}$, then $l(\theta) > l(f(\beta)) - 2\mathrm{BCC}(f) > \frac{\lambda + 1}{2} l(\beta)$. We call the constant $\kappa = \frac{4\mathrm{BCC}(f)}{\lambda - 1}$ the *legality threshold*. The legality threshold will have a key role in Chapter 5.

3.3 The Lipschitz Metric on $\mathrm{CV}'_n(\mathcal{A})$

In this section we define the Lipschitz metric on $\mathrm{CV}'_n(\mathcal{A})$. Basically, we think the wedge cycles as if they are infinitesimally small and we define the distance between $X, Y \in \mathrm{CV}'_n(\mathcal{A})$

only considering the Lipschitz constant of maps between the two graphs $\widehat{X}$ and $\widehat{Y}$ obtained respectively from X and Y collapsing the wedge cycles to points. Let $X = (\Gamma_1, \phi_1), Y = (\Gamma_2, \phi_2) \in \mathrm{CV}_n(\mathcal{A})$. Consider the graphs $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ given by collapsing the wedge cycles to points in Γ_1 and Γ_2 respectively. We call the points in $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ corresponding to the wedge cycles special points. Denote by $\mathrm{col}_m : \Gamma_m \to \widehat{\Gamma}_m$ the collapsing map, $m = 1, 2$. Let $\widehat{\phi}_m = \mathrm{col}_m \circ \phi$.

If $h : \Gamma_1 \to \Gamma_2$, $h \circ \phi_1 \sim \phi_2$ rel. C_i^j, for all i, j, then denote by $\widehat{h} : \widehat{\Gamma}_1 \to \widehat{\Gamma}_2$ the map obtained from h by collapsing the wedge cycles in Γ_1 and Γ_2 to points, sending special points to special points, and collapsing the edges corresponding to the pre-images of special points to points. Notice that $\widehat{h}$ is well-defined and $\widehat{h} \circ \widehat{\phi}_1 \sim \widehat{\phi_2}$. Let $\mathrm{Lip}(\widehat{h})$ be the Lipschitz constant of $\widehat{h}$, that is the smallest number among the possible values of K such that

$$d_{\widehat{\Gamma_2}}(\widehat{h}(p_1), \widehat{h}(p_2)) \leq K d_{\widehat{\Gamma_1}}(p_1, p_2),$$

for all points $p_1, p_2 \in \widehat{\Gamma_1}$. Define

$$L(X, Y) = \left\{ \mathrm{Lip}(\widehat{h}) \in \mathbb{R} \,|\, \widehat{h} \circ \widehat{\phi_1} \sim \widehat{\phi_2} \text{ rel. to the special points} \right\}.$$

Define the distance from $X = (\Gamma_1, \phi_1)$ to $Y = (\Gamma_2, \phi_2)$ to be:

$$d(X, Y) = \log \inf L(X, Y).$$

By Arzela-Ascoli, $L(X, Y)$ has a minimum and we call $\widehat{h} : \widehat{\Gamma_1} \to \widehat{\Gamma_2}$ an *optimal map* if $\mathrm{Lip}(\widehat{h}) = \inf L(X, Y)$. Moreover, it is enough to consider maps which are linear on edges. Indeed, suppose that $\widehat{h}$ is not linear on edges. Define $\widehat{h_1}$ in the following way: $\widehat{h}_1(v) = \widehat{h}(v)$ on every vertex v and sending an edge (v, w) to the immersed path $[\widehat{h}(v), \widehat{h}(w)]$ which is homotopic to $\mathrm{Im}(\widehat{h}|_{(v,w)})$ relatively to the endpoints and to the special points and parameterized at constant speed. It is clear that $\mathrm{Lip}(\widehat{h}_1) \leq \mathrm{Lip}(\widehat{h})$ and $\widehat{h_1}$ is homotopic to $\widehat{h}$ relatively to the special points. Therefore, we can restrict our attention to maps linear on edges. Obviously, $d(\cdot, \cdot)$ is not symmetric since it is not even symmetric in the case of outer space CV_n (see [24] or [1]).

Proposition 61. Let $X = (\Gamma_1, \phi_1)$, $Y = (\Gamma_2, \phi_2)$, $Z = (\Gamma_3, \phi_3)$ in $\mathrm{CV}'_n(\mathcal{A})$. We have:

1. $d(X, Y) \geq 0$ and $d(X, Y) = 0$ if and only if $X = Y$.

2. (Triangle inequality) $d(X, Z) \leq d(X, Y) + d(Y, Z)$.

3. $\mathrm{Out}(F_n; \mathcal{A})$ acts by isometries on $\mathrm{CV}'_n(\mathcal{A})$.

Proof. 1. Since the relative volume of each graph is 1 and if $d(X,Y) = \log(\mathrm{Lip}(\widehat{h}))$ then $\widehat{h}$ stretches all edges at most $\mathrm{Lip}(\widehat{h})$, we have

$$1 = \sum_{e \subset \widehat{\Gamma_2}} l(e) \leq \sum_{e \subset \mathrm{Im}(\widehat{h})} l(e) \leq \mathrm{Lip}(\widehat{h}) \sum_{e \subset \widehat{\Gamma_1}} l(e) = 1.$$

Therefore, $d(X,Y) = \log(\mathrm{Lip}(\widehat{h})) \geq 0$. If $d(X,Y) = 0$, then there is a (linear) map $\widehat{h}$ homotopic to the difference in marking with $\mathrm{Lip}(\widehat{h}) = 1$. We need to prove that $\widehat{h}$ is an isometry. First we will prove that $\widehat{h}$ is a bijection. Since $\widehat{h}$ is a homotopy equivalence and $\widehat{\Gamma_1}$, $\widehat{\Gamma_2}$ do not contain valence 1 vertices except for the special points, $\widehat{h}$ is surjective. Now, since $\mathrm{Lip}(\widehat{h}) = 1$, $\widehat{h}$ is onto and linear, and

$$\sum_{e \subset \widehat{\Gamma_1}} l(e) = \sum_{e \subset \widehat{\Gamma_2}} l(e) = 1,$$

we conclude that $\widehat{h}$ is an isometry from $\widehat{X}$ to $\widehat{Y}$. Because $\widehat{h}$ is homotopic to the difference in marking and the length of the edges in the wedge cycles is 0, we get $X = Y$ in $\mathrm{CV}'_n(\mathcal{A})$.

2. If $\widehat{h_1} : \widehat{\Gamma_1} \longrightarrow \widehat{\Gamma_2}$, $\widehat{h_2} : \widehat{\Gamma_2} \longrightarrow \widehat{\Gamma_3}$ are maps such that $\mathrm{Lip}(\widehat{h_1}) = L(X,Y)$ and $\mathrm{Lip}(\widehat{h_2}) = L(Y,Z)$, then $\widehat{h} = \widehat{h_2} \circ \widehat{h_1}$ is homotopic to $\widehat{\phi_3} \circ \widehat{\phi_1}^{-1}$ relatively to the special points, thus $\mathrm{Lip}(\widehat{h}) \geq \min L(X,Z)$. But $\mathrm{Lip}(\widehat{h_1})\mathrm{Lip}(\widehat{h_2}) \geq \mathrm{Lip}(\widehat{h}) \geq \min L(X,Z)$. Therefore,

$$d(X,Y) + d(Y,Z) = \log(\mathrm{Lip}(\widehat{h_1})) + \log(\mathrm{Lip}(\widehat{h_2})) \geq \log(\min L(X,Z)) = d(X,Z).$$

3. Obviously, $d(X \cdot \Phi, Y \cdot \Phi) = d(X,Y)$, for any $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$.

□

If α_X is an immersed loop in X, then it represents a conjugacy class α of $\pi_1(X, \phi(v))$. For every modified marked $(\mathcal{A}, n)$-graph Y, we shall denote by α_Y the immersed loop representing α in the graph Y. If α_X is not an immersed loop, we denote by $[\alpha_X]$ the immersed path representing the same conjugacy class. Recall that $\widehat{\alpha}$ is the path obtained by α collapsing the wedge cycles in Γ to (special) points. For a path γ_X in $X \in \mathrm{CV}'_n(\mathcal{A})$, the length of γ_X in X is denoted by $l(\gamma_X, X)$. For a conjugacy class γ in F_n, the length of the immersed loop representing γ in X will be denoted by $l(\gamma, X)$. For a path $\widehat{\alpha}$, we denote by $l(\widehat{\alpha}, X)$ the length of the path in $\widehat{X} = (\widehat{\Gamma}, \widehat{\phi})$.

For two points $X, Y \in \mathrm{CV}'_n(\mathcal{A})$ and an immersed loop α, denote the stretch of $\widehat{\alpha}$ from $\widehat{X}$ to $\widehat{Y}$ by $\mathrm{St}_{\widehat{\alpha}}(X,Y) = \frac{l(\widehat{h(\alpha)}, Y)}{l(\widehat{\alpha}, X)}$. Let $S(X,Y) = \{\mathrm{St}_{\widehat{\alpha}}(X,Y)\}$.

Definition 62. We say that a loop α in X is a *candidate* if $\widehat{\alpha}$ is one of the following:

1. an arc such that the first and the last vertices are special points;

2. an embedded circle with an arc attached such that the last vertex of the arc (not in common with the embedded circle) is a special point;

3. an embedded circle;

4. a figure eight, i.e., a wedge of two embedded circles that intersect in one point and $\widehat{\alpha}$ crosses each circle once and does not cross any edge outside these two embedded circles;

5. a barbell, i.e., $\widehat{\alpha}$ is a concatenation of $\gamma_1\gamma_2\gamma_3\overline{\gamma_2}$ where γ_1 and γ_3 are disjoint embedded circles and γ_2 is an embedded path which intersects γ_i, $i = 1, 3$ in exactly one point which is one of its endpoints.

Denote by $\mathrm{Can}(X)$ the set of candidates of X.

Note that there are a finite number of such $\widehat{\alpha}$ because there are a finite number of edges in each graph. Therefore there are a finite number of equivalence classes such that two candidates α_1 and α_2 are equivalent if $\widehat{\alpha_1} = \widehat{\alpha_2}$. Let $\{\alpha\}$ denote the class of α. However, if α is passing through at least a wedge cycle, $\{\alpha\}$ contains an infinite number of candidates. For example, let $\alpha = C_1 \cdot e_1 \cdots e_s$ be a candidate, where C_1 is a cycle and e_i is an edge in $\widehat{\Gamma}$ for $i = 1, \ldots, s$. Consider α_p as the loop obtained by going around the cycle C_1 p times and then following the path $e_1 \cdots e_s$. Obviously, α_p is a candidate and $\alpha_p \in \{\alpha\}$.

Definition 63. Let $X = (\Gamma_1, \phi_1)$, $Y = (\Gamma_2, \phi_2) \in \mathrm{CV}'_n(\mathcal{A})$ and $\widehat{h} : \widehat{\Gamma_1} \to \widehat{\Gamma_2}$ be an optimal (linear) map. The *tension subgraph* $\Gamma_h \subset \widehat{\Gamma_1}$ with respect to $\widehat{h}$ is the union of edges on which the slope of $\widehat{h}$ equals $\min L(X, Y)$.

Now we prove that the optimal Lipschitz constant of a map $\widehat{h} : \widehat{\Gamma_1} \to \widehat{\Gamma_2}$ is equal to the stretch factor of the maximally stretched restriction $\widehat{\alpha}$ of a loop α in X.

Theorem 64. We have

$$\min L(X, Y) = \max S(X, Y).$$

Moreover, there is $\alpha \in \mathrm{Can}(X)$ such that

$$d(X, Y) = \log\Big(\frac{l(\widehat{h(\alpha)}, Y)}{l(\widehat{\alpha}, X)}\Big).$$

The proof is similar to the proof of Proposition 2.3 in [1].

Proof. Let α be a conjugacy class represented in X by the immersed loop α_X. Since $h(\alpha_X)$ that represents α in Y might not be immersed, we denote by $[h(\alpha_X)]$ the path homotopic to $h(\alpha_X)$ relatively to the wedge cycles. Then

$$l(\widehat{\alpha}, Y) = l(\widehat{[h(\alpha_X)]}, Y) \leq l(\widehat{h(\alpha_X)}, Y) \leq \mathrm{Lip}(\widehat{h}) l(\widehat{\alpha}, X). \tag{3.1}$$

We deduce that $\max S(X,Y) \leq \min L(X,Y)$. Notice that we get equality in (3.1) if and only if all of the edges which $\widehat{\alpha}_X$ crosses are stretched by $\mathrm{Lip}(\widehat{h})$ and $\widehat{[h(\alpha_X)]} = \widehat{h(\alpha_X)}$. The goal is to show that a map $\widehat{h}$ with such a loop α exists.

Given a map $\widehat{h} : \widehat{\Gamma_1} \longrightarrow \widehat{\Gamma_2}$ which is linear on edges, let Γ_h be the tension subgraph in $\widehat{\Gamma_1}$ respect to $\widehat{h}$. The map $\widehat{h}$ induces a train track structure on Γ_h (as defined in Section 3.2). Let $\widehat{h}$ be a map such that $\mathrm{Lip}(\widehat{h}) = \inf L(X,Y)$, linear on edges and so that Γ_h is smallest among all the maps $\widehat{h_i}$ such that $\mathrm{Lip}(\widehat{h_i}) = L(X,Y)$. We claim that there are at least two gates at each vertex of Γ_h that is not a special point. By contradiction, suppose that there is a vertex v (not a special point) where Γ_h contains only one gate at v. Let $S_2 = \max\{\widehat{h}'(e) | e \nsubseteq \Gamma_h,\ e \subseteq \widehat{\Gamma_1}\}$, l_{short} be the length of shortest edge in X which is not trivial, and

$$\varepsilon = \frac{1}{2}(\mathrm{Lip}(\widehat{h}) - S_2) \cdot l_{\text{short}}.$$

Define a map $\widehat{h}_1$ by $\widehat{h}_1(u) = \widehat{h}(u)$ for all vertices $u \neq v$. To define $\widehat{h}_1(v)$ take an edge $e \subset \Gamma_h$ adjacent to v, let $\widehat{h}_1(v)$ be the point on the germ defined by $\widehat{h}(e)$ (that does not depend on e since there is only one gate at v) a distance $< \varepsilon$ away from $\widehat{h}(v)$. Define $\widehat{h_1}$ to be homotopic to $\widehat{h}$ rel. to the special points, and linear on edges.

Since $l(\psi(e), Y) \leq S_2 \cdot l(e, X) + 2\varepsilon < \mathrm{Lip}(\widehat{h}) l(e, X)$, we have $\widehat{h}_1'(e) < \mathrm{Lip}(\widehat{h})$. Therefore, either $\mathrm{Lip}(\widehat{h}_1) = \mathrm{Lip}(\widehat{h})$ with $\Gamma_{h_1} \subsetneq \Gamma_h$, or $\mathrm{Lip}(\widehat{h}_1) < \mathrm{Lip}(\widehat{h})$. In both cases we have a contradiction. This concludes the proof of the claim. Consider a legal path $\alpha_X \in X$ which intersects itself twice and $\widehat{\alpha}_X \in \Gamma_h$. Such a path will contain a legal subloop. We construct a candidate loop in the following way. We start from a vertex and we follow the path till it closes up for the first time. If that loop is legal and not contained in the wedge cycles, we are done. Otherwise, we keep following the path. There are different cases. However, all the possibilities are contained in the list of candidates.

Formally, parameterize α_X so that $\alpha : [0, M] \longrightarrow \Gamma$ and we can suppose $\alpha(0) = \alpha(t_1)$, where $0 < t_1 \leq M$. Otherwise, if $\alpha(t_1) = \alpha(t_2)$, with $0 < t_1 < t_2 \leq M$, then we consider the path $\alpha_1(t) = \alpha(t - t_1)$, $\alpha_1 : [0, M_1] \longrightarrow \Gamma$. Let $D_+\widehat{\alpha}(0)$ be the first oriented edge in $\widehat{\alpha}$ at time 0 and let $D_-\widehat{\alpha}(t_1)$ be the last oriented (with opposite orientation) edge in $\widehat{\alpha}$ at time t_1. If the turn $\{D_+\widehat{\alpha}(0), D_-\widehat{\alpha}(t_1)\}$ is legal and $\widehat{\alpha}([0, t_1])$ is not a special point, then $\alpha([0, t_1])$ is a legal loop and α is a candidate loop of type 3.

If $\{D_+\widehat{\alpha}(0), D_-\widehat{\alpha}(t_1)\}$ is illegal and $\widehat{\alpha}(0)$ is not a special point, then, by the previous claim, there is another gate at $\widehat{\alpha}(0)$ in Γ_h. Extend α legally to cross this gate and continue until there are $t_2 < t_3$ so that $\alpha(t_2) = \alpha(t_3)$. We are in one of the following cases:

1. If $t_2 < t_1 < t_3$, then either
   - $\alpha([t_2, t_3])$ is a legal loop, and hence $\alpha([t_2, t_3])$ is a candidate loop of type 3 or
   - $\alpha([0, t_2]) \cup \alpha^{-1}([t_1, t_3])$ is a legal loop and α is a candidate loop of type 3.
2. If $t_1 \leq t_2 < t_3$, then either
   - $\alpha([t_2, t_3])$ is a legal loop, $\widehat{\alpha}([t_2, t_3])$ is not a special point, and $\alpha([t_2, t_3])$ is a candidate loop of type 3 or
   - $\alpha([0, t_3]) \cup \alpha^{-1}([t_1, t_2])$ is a legal loop, $\alpha(t_1) = \alpha(t_2)$, $\widehat{\alpha}([t_2, t_3])$ is not a special point, and therefore α is a candidate loop of type 4 or
   - $\alpha([0, t_3]) \cup \alpha^{-1}([t_1, t_2])$ is a legal loop, $\alpha(t_1) \neq \alpha(t_2)$, $\widehat{\alpha}([t_2, t_3])$ is not a special point, and therefore α is a candidate loop of type 5 or
   - $\alpha([0, t_3]) \cup \alpha^{-1}([t_1, t_2])$ is a legal loop, $\alpha(t_1) \neq \alpha(t_2)$, $\widehat{\alpha}([t_2, t_3])$ is a special point, and therefore α is a candidate loop of type 2.

If $\widehat{\alpha}([0, t_1])$ is a special point, then extend α legally outside the wedge cycles until there are $t_2 < t_3$ so that $\alpha(t_2) = \alpha(t_3)$. We are in one of the following cases:

1. $\alpha([t_2, t_3])$ is a special point, therefore $\widehat{\alpha}$ is a tree and α is a candidate loop of type 1.
2. If $t_2 < t_1 < t_3$, then either
   - $\alpha([t_2, t_3])$ is a legal loop, and hence $\alpha([t_2, t_3])$ is a candidate loop of type 3 or
   - $\alpha([0, t_2]) \cup \alpha^{-1}([t_1, t_3])$ is a legal loop and α is a candidate loop of type 3.
3. If $t_1 \leq t_2 < t_3$, then either
   - $\alpha([t_2, t_3])$ is a legal loop, and $\alpha([t_2, t_3])$ is a candidate loop of type 3 or
   - $\alpha([0, t_3]) \cup \alpha^{-1}([t_1, t_2])$ is a legal loop, $\alpha(t_1) \neq \alpha(t_2)$, and therefore α is a candidate loop of type 2;
   - $\alpha([0, t_3]) \cup \alpha^{-1}([t_1, t_2])$ is a legal loop, $\alpha(t_1) = \alpha(t_2)$, and therefore α is a candidate loop of type 3.

Therefore, there is a maximally stretched restriction of a loop α with stretch constant equal to the Lipschitz constant of $\widehat{h}$ and α is a candidate loop. $\square$

We conclude this section with an interesting result.

Lemma 65. For any $X \in \mathrm{CV}'_n(\mathcal{A})$ and $D > 0$, the set

$$T = \{X \cdot \Phi \,|\, \Phi \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA},\, d(X, X \cdot \Phi) = D\}$$

is discrete.

Proof. By contradiction, there exist a sequence $\Phi_k \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ such that $d(X \cdot \Phi_k, X \cdot \Phi) \leq \frac{1}{k}$ and $d(X, X \cdot \Phi_k) = D$. Given X, we know that there are finitely many classes of candidates for $X \to X \cdot \Phi \circ \Phi_k^{-1}$. Therefore, there exist only finitely many $D_k = d(X \cdot \Phi_k, X \cdot \Phi)$. Hence, there is N such that $\Phi_N = \Phi$. $\square$

3.4 The Relative Train Track Theorem

We will use the Lipschitz distance defined on the modified relative outer space to prove the Relative Train Track Theorem.

Theorem 66. (Relative Train Track Theorem) Every irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$ of infinite order has a topological representative which is a train track map.

We will use the same approach as in [4]. First we will prove a Bers' like theorem about the classification of relative outer automorphisms. Let $[\Phi] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ and $\widehat{\Phi}$ be the map obtained from Φ by collapsing the wedge cycles to special points.

Definition 67. Let $[\Phi] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$. The map

$$[\widetilde{\Phi}] : \mathrm{CV}'_n(\mathcal{A}) \to [0, \infty), \quad [\widetilde{\Phi}](X) = d(X, X \cdot \Phi)$$

is called *displacement function*.

Notice that the displacement function is well-defined since if $[\Phi_1] = [\Phi_2]$, then $d(X, X \cdot \Phi_1) = d(X, X \cdot \Phi_2)$.

Definition 68. For $\theta > 0$, the *θ-thick part* of $\mathrm{CV}'_n(\mathcal{A})$ is

$$\mathrm{CV}'_n(\mathcal{A})^\theta = \{X \in \mathrm{CV}'_n(\mathcal{A}) \,|\, l(\widehat{\alpha}, X) \geq \theta \text{ for all } \alpha \subset X\}.$$

Proposition 69. $\mathrm{CV}'_n(\mathcal{A})^\theta/\mathrm{Out}(F_n; \mathcal{A})$ is compact.

Proof. As remarked in Section 3.1, the space $\mathrm{CV}'_n(\mathcal{A})$ is union of open simplices and hence $\mathrm{CV}'_n(\mathcal{A})^\theta$ is the union of truncated simplices. Indeed, consider a simplex σ (with missing faces) in $\mathrm{CV}'_n(\mathcal{A})$ and restrict σ to the thick part. Denote by σ' such a restriction. As in the case of outer space, σ' is a closed truncated simplex because the length of a loop is small only close to a missing face. Since there are only finitely many orbits of open simplices, $\mathrm{CV}'_n(\mathcal{A})^\theta/\mathrm{Out}(F_n;\mathcal{A})$ is compact. □

Theorem 70. Let $[\Phi] \in \mathrm{Out}(F_n;\mathcal{A})/\mathrm{KA}$. Then there are three possibilities for $[\widetilde{\Phi}]$:

- (elliptic) $\inf[\widetilde{\Phi}] = 0$ and it is realized. In this case, there exists k such that $\Phi^k \in \mathrm{KA}$.
- (hyperbolic) $\inf[\widetilde{\Phi}] > 0$ and it is realized. In this case there exists train track map $\phi : X \to X \cdot \Phi$, that is, an optimal map which sends each edge to a legal path and legal turns to legal turns.
- (parabolic) $\inf[\widetilde{\Phi}]$ is not realized. In this case Φ is reducible.

Proof.
- If $\inf[\widetilde{\Phi}] = 0$ and it is realized, $\widehat{\Phi}$ has a fixed point, i.e., $d(X, X \cdot \Phi) = 0$, for some $X = (\Gamma, \psi) \in \mathrm{CV}'_n(\mathcal{A})$. We have $\widehat{\psi} \circ \widehat{\Phi} \simeq \widehat{f} \circ \widehat{\psi}$ relative to the special points, for an isometry $\widehat{f} : \widehat{\Gamma} \to \widehat{\Gamma}$. Notice that the isometries of $\widehat{\Gamma}$ have finite order. Therefore, there exists $k > 0$ such that $\widehat{\Phi}^k$ is homotopic to the identity. Hence $\widehat{\Phi}$ has finite order. In other words, the relative transition matrix is a permutation matrix and there exists k such that $\Phi^k \in \mathrm{KA}$.
- If $\inf[\widetilde{\Phi}] > 0$ and it is realized, then suppose that $\inf[\widetilde{\Phi}]$ is realized on $X = (\Gamma, \psi) \in \mathrm{CV}'_n(\mathcal{A})$. Let $\log(\lambda) = d(X, X \cdot \Phi) = \inf[\widetilde{\Phi}] > 0$. Thus, $\lambda > 1$. Let $\phi : X \to X \cdot \Phi$ be an optimal map and $\Gamma_\phi \subset \widehat{\Gamma}$ be the tension subgraph respect to ϕ with its train track structure.

  Claim 71. After an arbitrarily small perturbation of X that preserves the condition that $d(X, X\cdot\Phi) = \log(\lambda)$, there is an optimal map $\phi : X \to X\cdot\Phi$ such that $\widehat{\phi}(\Gamma_\phi) \subset \Gamma_\phi$ and $\widehat{\phi}$ sends edges of Γ_ϕ to legal paths. Moreover, $\widehat{\phi}$ sends legal turns to legal turns.

  Proof. Fix an optimal map $\phi : X \to X \cdot \Phi$. Suppose that $\widehat{\phi}(\Gamma_\phi) \nsubseteq \Gamma_\phi$ and let e be an edge of Γ_ϕ with $\widehat{\phi}(e) \nsubseteq \Gamma_\phi$. If we perturb the metric on $\widehat{\Gamma}$ by scaling on the wedges of Γ_ϕ by a constant greater than 1 and by a constant less than 1 in its complement in such a way that the relative volume of the new graph $\widehat{\Gamma}_1$ that we get changing the metric is 1 and the tension subgraph $(\Gamma_1)_{\phi_1}$ in $\widehat{\Gamma}_1$ with respect to $\phi_1 : X_1 \to X_1 \cdot \Phi$ (obtained by ϕ) is contained in Γ_ϕ and does not contain e. Note that the slope on some edge must be λ by the minimality assumption. Iterating this process, we finally get a perturbation of X where $\widehat{\phi}(\Gamma_\phi) \subset \Gamma_\phi$. We can assume that there are no vertices in Γ_ϕ (that are not special points) with one gate. Otherwise, we can proceed as in Theorem 64 and find a smaller tension graph. Suppose that $\widehat{\phi}$ maps an edge e of Γ_ϕ over an illegal turn. We can fold the illegal turn and perturb the map to eliminate vertices (not special points) with one gate. Now, suppose that $\widehat{\phi}$ maps a legal turn to an illegal turn. Perturb by folding the illegal turn. This converts the legal turn to an illegal turn. This move lowers the number

$$\sum_v [\max\{0, G(v) - 2\}],$$

where $G(v)$ is the number of gates at v (not special point). If Γ_ϕ does not have all vertices, that are not special points, with ≥ 2 gates, perturb as before to get a smaller Γ_ϕ. If we define the complexity of Γ_ϕ has $(\text{rank}H_1(\Gamma_\phi), -\text{rank}H_0(\Gamma_\phi))$ with the lexicographic order, we see that perturbing the map as before we can only decrease the complexity. Therefore, the process has to stop in a finite number of steps. In conclusion, we have ϕ and Γ_ϕ that we were looking for. $\square$

- If $\inf[\widetilde{\Phi}]$ is not realized, then consider a sequence $\{X_k = (\Gamma_k, \psi_k)\}$ of points in the modified relative outer space such that

$$d(X_k, X_k \cdot \Phi) \xrightarrow[k\to\infty]{} D = \inf[\widetilde{\Phi}],$$

but D is not realized.

Claim 72. For any $\theta > 0$, there are only finitely many X_k with $X_k \in \text{CV}'_n(\mathcal{A})^\theta$.

Proof. By contradiction, suppose that there are infinitely many X_k with $X_k \in \text{CV}'_n(\mathcal{A})^\theta$. After passing to a subsequence, assume that $X_k \in \text{CV}'_n(\mathcal{A})^\theta$ for every k. By Proposition 69, after passing to a subsequence, there are $\Upsilon_k \in \text{Out}(F_n; \mathcal{A})$ such that $X_k \cdot \Upsilon_k \xrightarrow[k\to\infty]{} X_\infty$. Therefore, we have

$$d(X_\infty \cdot \Upsilon_k^{-1}, X_\infty \cdot \Upsilon_k^{-1} \circ \Phi) \leq d(X_\infty \cdot \Upsilon_k^{-1}, X_k) + d(X_k, X_k \cdot \Phi) + d(X_k \cdot \Phi, X_\infty \cdot \Upsilon_k^{-1} \circ \Phi) =$$

$$= d(X_\infty, X_k \cdot \Upsilon_k) + d(X_k, X_k \cdot \Phi) + d(X_k \cdot \Upsilon_k, X_\infty).$$

Since $d(X_k \cdot \Upsilon_k, X_\infty) \xrightarrow[k\to\infty]{} 0$, $d(X_\infty, X_k \cdot \Upsilon_k) \xrightarrow[k\to\infty]{} 0$, and $d(X_k, X_k \cdot \Phi) \xrightarrow[k\to\infty]{} D$, we have

$$d(X_\infty \cdot \Upsilon_k^{-1}, X_\infty \cdot \Upsilon_k^{-1} \circ \Phi) \xrightarrow[k\to\infty]{} D,$$

and hence

$$d(X_\infty, X_\infty \cdot \Upsilon_k^{-1} \circ \Phi \circ \Upsilon_k) \xrightarrow[k\to\infty]{} D.$$

Since there are finitely many classes of candidates for $X_\infty \to X_\infty \cdot \Upsilon_k^{-1} \circ \Phi \circ \Upsilon_k$, there are only finitely many distances

$$D_k = d(X_\infty, X_\infty \cdot \Upsilon_k^{-1} \circ \Phi \circ \Upsilon_k) \leq D + 1.$$

Therefore, after passing to a subsequence, $\{D_k\}$ is constant and

$$d(X_\infty, X_\infty \cdot \Upsilon_k^{-1} \circ \Phi \circ \Upsilon_k) = D,$$

which contradicts the hypothesis that the infimum is not realized. This concludes the proof of the claim. □

For $\varepsilon > 0$, let Γ^ε be the subgraph of $\widehat{\Gamma}$ union of all essential loops of length $\leq \varepsilon$. This is always a graph with vertices (not special points) of valence ≥ 2 (possibly empty). There is $\varepsilon_n > 0$ such that for any $X = (\Gamma, \psi) \in \mathrm{CV}'_n(\mathcal{A})$, the subgraph Γ^{ε_n} is always proper. Moreover, there is a bound B_n to the length of any chain of proper subgraphs with valence ≥ 2 vertices (not special points) and valence ≥ 1 special points.

Claim 73. For large k, any optimal map $X_k \to X_k \cdot \Phi$ leaves a nonempty proper subgraph Δ with valence ≥ 2 vertices of $\widehat{\Gamma_k}$ invariant up to homotopy relative to the wedge cycles.

Proof. Let $\theta = \frac{\varepsilon_n}{e^{(D+1)^{B_n}}}$. We know that eventually $X_k \notin \mathrm{CV}'_n(\mathcal{A})^\theta$. Choose k large enough so that $d(X_k, X_k \cdot \Phi) < D + 1$. Let $\delta_i := \frac{\varepsilon_n}{e^{(D+1)^i}}$, for $i = 1, 2, \dots, B_n$. Then, we have a chain

$$\Gamma_k^{\delta_{B_n}} \subseteq \cdots \subseteq \Gamma_k^{\delta_0},$$

where each element is homotopic (rel. to the wedge cycles) to a nonempty proper subgraph. The length of the chain is $B_n + 1$, so there exists i such that $\Gamma_k^{\delta_i}$ and $\Gamma_k^{\delta_{i+1}}$ have the same subgraph Δ with valence ≥ 2 vertices. By definition, an optimal map must send $\Gamma_k^{\delta_{i+1}}$ into $\Gamma_k^{\delta_i}$, so Δ is a proper invariant nonempty subgraph in $\widehat{\Gamma_k}$ up to homotopy relatively to the wedge cycles. □

This claim concludes the proof of the theorem. □

Remark 74. The proof of Theorem 70 is similar to the proof of the main result in [4]. However, notice that in the parabolic case we could not use the fact that the space is locally compact since it is not true for the modified relative outer space. Instead, we reach to the same conclusion basing the argument on the finiteness of the classes of candidates.

Example 75. If $n = 2$ and $A =< a >$, any $\Phi \in \mathrm{Out}(F_2; A)$ is elliptic (see Example 49).

Corollary 76. Let X realize $\inf[\widetilde{\Phi}] = \log(\lambda) > 0$. Then, for any $m = 1, 2, \ldots$, $d(X, X \cdot \Phi^m) = m\,\log(\lambda)$.

Proof. After perturbing, as in Theorem 70, if α is a legal loop, then $\phi(\alpha)$ is a legal loop, and the length of $\widehat{\phi}(\widehat{\alpha})$ is equal to $\lambda l(\widehat{\alpha})$. Dy induction (after perturbing), the length of $\widehat{\phi}^m(\widehat{\alpha})$ is equal to $\lambda^m l(\widehat{\alpha})$. By continuity, this is also true before perturbing. $\square$

Remark 77. If $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$, $\phi : X \to X \cdot \Phi$ is an optimal map with $\widehat{\phi}(\Gamma_\phi) \subseteq \Gamma_\phi$, $\widehat{\phi}$ sends edges of Γ_ϕ to legal paths, and $\widehat{\phi}$ preserves legal turns, then Φ achieves the minimum at X. Indeed, if $m > 0$ and $X_1 = (\Gamma_1, \psi_1) \in \mathrm{CV}'_n(\mathcal{A})$, then

$$m \cdot d(X, X \cdot \Phi) = d(X, X \cdot \Phi^m) \leq d(X, X_1) + d(X_1, X_1 \cdot \Phi^m) +$$

$$+ d(X_1 \cdot \Phi^m, X \cdot \Phi^m) \leq d(X, X_1) + m \cdot d(X_1, X_1 \cdot \Phi) + d(X_1, X).$$

Therefore,

$$d(X, X \cdot \Phi) \leq \frac{d(X, X_1)}{m} + d(X_1, X_1 \cdot \Phi) + \frac{d(X_1, X)}{m}.$$

and, as $m \to \infty$, we have

$$d(X, X \cdot \Phi) \leq d(X_1, X_1 \cdot \Phi).$$

Remark 78. From Corollary 76 it follows that if $\mathrm{CV}'_n(\mathcal{A})$ contains a hyperbolic element (e.g. $n - \sum_{i=1}^{k} s(i) > 1$), then the diameter of $\mathrm{CV}'_n(\mathcal{A})$ in infinite.

We proved the weak version of the Relative Train Track Theorem in which we did not require that vertices go to vertices. Let's complete the proof of the Relative Train Track Theorem in the following way. Let v be a vertex of $\widehat{\Gamma}$. Notice that special points go to special points. Therefore we can suppose that v is not a special point. Let ϕ_t, $t \in [0, 1]$, be a homotopy of $\phi = \phi_0$ rel. to the wedge cycles that moves each $\phi(v)$ that is not a vertex to an endpoint of the edge containing $\phi(v)$. In this choice we give the priority to endpoints that are not special points. We also insist that during the homotopy the order of the images of the vertices in the same edge do not change until the very last moment when several images of vertices may arrive at the same vertex. Obviously, the images of legal loops under ϕ_t are still legal loops. There might be edges that maps to points under ϕ_1. In this case, we collapse the edges iteratively. We obtain a new map $\phi' : \Gamma' \to \Gamma'$ representing

Φ. Any legal loop in Γ induces a loop in Γ' whose ϕ'-iterates are immersed. Notice that if α is a loop in Γ with positive length, then the iterates of α cross every edge of $\widehat{\Gamma}$. Therefore, iterated images of edges are immersed. Let $\Delta \subset \widehat{\Gamma}$ be a core graph. If d_1 and d_2 are two directions at $v \in \Delta$, let $d_1 \sim d_2$ if $(\phi')^k_* d_1 = (\phi')^k_* d_2$ for some $k \geq 1$. This defines a train track structure on Γ' (see Section 3.2).

3.5 Tangent Spaces in $\mathrm{CV}'_n(\mathcal{A})$

The purpose of this section is to prove that the Lipschitz metric is almost-symmetric (see Corollary 90). We will use the same approach as in [2]. Consider a modified marked metric $(\mathcal{A}, n)$-graph (Γ, ϕ, l). Let $\Sigma_{\widehat{\Gamma}}$ be the simplex with missing faces obtained by varying the length of the edges in $\widehat{\Gamma}$. Let ℓ be a metric in $\Sigma_{\widehat{\Gamma}}$. We define the tangent space

$$T_\ell(\Sigma_{\widehat{\Gamma}}) = \{\tau : E(\widehat{\Gamma}) \longrightarrow \mathbb{R} \mid \sum_{e \in E(\widehat{\Gamma})} \tau(e) = 0\},$$

where $E(\widehat{\Gamma})$ is the set of edges in $\widehat{\Gamma}$. If ℓ, ℓ' are two points in $\Sigma_{\widehat{\Gamma}}$, the natural identification between $T_\ell(\Sigma_{\widehat{\Gamma}})$ and $T_{\ell'}(\Sigma_{\widehat{\Gamma}})$ leads to a product decomposition

$$T(\Sigma_{\widehat{\Gamma}}) \cong \Sigma_{\widehat{\Gamma}} \times \mathbb{R}^{N-1}$$

of the total tangent space, where N is the number of edges of $\widehat{\Gamma}$.

Definition 79. A tangent vector $\tau \in T_\ell(\Sigma_{\widehat{\Gamma}})$ is *integrable* if $\tau(e) < 0$ implies $\ell(e) > 0$ for all $e \in E(\widehat{\Gamma})$. In that case we have the path $\ell + t\tau \in \Sigma_{\widehat{\Gamma}}$ for small $t \geq 0$.

Notice that if $X' = (\Gamma, \phi', l') \in \mathrm{CV}'_n(\mathcal{A})$ has $\widehat{\phi}' = \widehat{\phi}$ and $l' = l$, then the tangent spaces are equal. If Γ' is obtained from Γ by collapsing a forest in $\widehat{\Gamma}$, then we have natural inclusions $\Sigma_{\widehat{\Gamma'}} \subset \Sigma_{\widehat{\Gamma}}$ and $T(\Sigma_{\widehat{\Gamma'}}) \subset T(\Sigma_{\widehat{\Gamma}})$ given by considering metrics on Γ that vanish on the forest in $\widehat{\Gamma}$.

The set of integrable vectors in $T_\ell(\Sigma_{\widehat{\Gamma}})$ is a closed convex cone, i.e., $v, w \in T_\ell(\Sigma_{\widehat{\Gamma}})$ implies $tv + sw \in T_\ell(\Sigma_{\widehat{\Gamma}})$, for all $t, s \in [0, \infty)$.

Proposition 80 ([2] Proposition 6)**.**

1. Let $\tau \in T_\ell(\Sigma_{\widehat{\Gamma}})$ be an integrable vector. Then there is a candidate loop α in Γ such that

$$d((\Gamma, \phi, \ell), (\Gamma, \phi, \ell + t\tau)) = \log \frac{\widehat{(\ell + t\tau)}(\widehat{\alpha})}{\widehat{\ell}(\widehat{\alpha})}$$

for all sufficiently small $t \geq 0$.

2. $\lim_{t \longrightarrow 0^+} \frac{d((\Gamma, \phi, \ell), (\Gamma, \phi, \ell + t\tau))}{t} = \frac{\tau(\widehat{\alpha})}{\widehat{\ell}(\widehat{\alpha})}$ for the loop α in item (1).

3. The set of integrable vectors in $T_\ell(\Sigma_{\widehat{\Gamma}})$ is a finite union of closed convex cones $B_1, B_2, \cdots, B_N$ such that for any B_i there is a candidate loop α_i that realizes the distance $d((\Gamma,\phi,\ell),(\Gamma,\phi,\ell+t\tau))$ for any $\tau \in B_i$ and small $t \geq 0$.

The proof of this proposition is similar to the proof of the analog for outer space in [2]. The only difference consist in considering

$$\frac{\widehat{(\ell+t\tau)}(\widehat{\alpha})}{\widehat{\ell}(\widehat{\alpha})} \geq \frac{\widehat{(\ell+t\tau)}(\widehat{\beta})}{\widehat{\ell}(\widehat{\beta})}$$

instead of

$$\frac{(\ell+t\tau)(\alpha)}{\ell(\alpha)} \geq \frac{(\ell+t\tau)(\beta)}{\ell(\beta)}.$$

Now, our goal is to define a quasi-symmetric norm on $T(\Sigma_{\widehat{\Gamma}})$. Let's start with the following definition.

Definition 81. Let $\tau \in T_\ell(\Sigma_{\widehat{\Gamma}})$. Define

$$\|(\ell,\tau)\|^L = \sup\left\{ \frac{\tau(\widehat{\alpha})}{\widehat{\ell}(\widehat{\alpha})} \,\middle|\, \alpha \text{ is a loop in } \Gamma \text{ s.t. } \widehat{\ell}(\widehat{\alpha}) > 0 \right\}.$$

Thus we have an (asymmetric) norm for the Lipschitz metric. The problem is that the norm is not even quasi-symmetric, since it is not quasi-symmetric in the case of outer space (see [2]). We aim to define a new norm on $T_\ell(\Sigma_{\widehat{\Gamma}})$ which is quasi-symmetric. The idea is to correct $\|\cdot\|^L$ by adding the directional derivative of a function that we are going to define. First consider the relative homology $H_1(\Gamma, \bigcup \mathbb{B}_j; \mathbb{Z}_2)$. Let a be a nontrivial homology class in $H_1(\Gamma, \bigcup \mathbb{B}_j; \mathbb{Z}_2)$ (that is, not representing an element in a wedge cycle). Since the homology classes corresponding to the wedge cycles will be fixed, we are only interested in the homology classes not corresponding to the wedge cycles. By $\widehat{\ell}(\widehat{a})$ denote the minimal $\widehat{\ell}(\widehat{\alpha})$ where α ranges over the class of loops in a and $\alpha_1 \sim \alpha_2$ if collapsing the wedge cycles to special points we have $\widehat{\alpha_1} = \widehat{\alpha_2}$. Since there are only finitely many classes of candidates (see Section 3.3), this minimum exists, and it is realized in the classes of candidates $\alpha_1, \ldots, \alpha_k$.

Proposition 82 ([2] Proposition 9)**.** For each non-trivial $a \in H_1(\Gamma, \bigcup \mathbb{B}_j; \mathbb{Z}_2)$ there are finitely many classes of loops $\alpha_1, \ldots, \alpha_k$ so that $\widehat{\ell}(\widehat{a})$ is realized by some $\widehat{\alpha}_i$ for all $\ell \in \Sigma_{\widehat{\Gamma}}$. Moreover, if α is a loop then, for all $\ell \in \Sigma_{\widehat{\Gamma}}$, α is the shortest class of loops representing $[\alpha]$.

Remark 83. Notice that in [2] the homology group is $H_1(\Gamma; \mathbb{Z}_2)$. Indeed, in that case $\bigcup \mathbb{B}_j = \emptyset$.

Let $\Gamma_i \longrightarrow \Gamma$, $i = 1, 2, \cdots, 2^{n-\sum s(i)-1}$ be the collection of all nontrivial double covers of Γ. Notice that the lift of an edge in a wedge cycle has length 0 in Γ_i. Any $\ell \in \Sigma_{\widehat{\Gamma}}$ induces a metric ℓ_i on each Γ_i by pulling back, and likewise any tangent vector $\tau \in T_\ell(\Sigma_{\widehat{\Gamma}})$ lifts to a tangent vector in $T_{\ell_i}(\Sigma_{\widehat{\Gamma}_i})$. Let $\widetilde{\mathbb{B}_j}$ be the lift of $\mathbb{B}_j$. If $a \in H_1(\Gamma_i, \bigcup \widetilde{\mathbb{B}_j}; \mathbb{Z}_2)$ is a given homology class not contained in a lift of a wedge cycle, denote by $\ell_i(\widehat{a})$ the length of a shortest loop in $\widehat{\Gamma}_i$ equipped with ℓ_i that represents a.

Lemma 84. If α is a candidate in Γ, then there exists a double cover $\Gamma_i \longrightarrow \Gamma$, and a lift $\tilde{\alpha}$ of α so that $\tilde{\alpha}$ is the unique shortest class of loops in its (nontrivial) homology class.

Proof. We will show that we can arrange that $\tilde{\alpha}$ is shortest in its homology class. If $\widehat{\alpha}$ is an embedded circle, then any double cover to which α lifts works. If $\widehat{\alpha}$ is a figure eight or a barbell, take the double cover by cutting and regluing along two points, one in each embedded circle of $\widehat{\alpha}$. If $\widehat{\alpha}$ is an arc or an embedded circle of positive length with possibly an arc attached, then any double cover to which α lifts works. □

Definition 85. Let

$$N(\ell, \tau) = -\sum_{\Gamma_i} \sum_{a \in H_1(\Gamma_i, \bigcup \widetilde{\mathbb{B}_j}; \mathbb{Z}_2) \setminus \{0\}} \frac{\max \tau(\widehat{\alpha})}{\widehat{\ell}(\widehat{a})}, \tag{3.2}$$

where maximum is taken over all loops α in Γ_i that realize $\widehat{\ell}(\widehat{a})$ and does not represent an element in a wedge cycle. Define the new norm by

$$\|(\ell, \tau)\|^N = \|(\ell, \tau)\|^L + \frac{1}{K+1} N(\ell, \tau), \tag{3.3}$$

where K is the number of summands in (3.2).

As in [2], $\|\cdot\|^N$ is a (nonsymmetric) norm. Define the map $\Psi : \Sigma_{\widehat{\Gamma}} \longrightarrow \mathbb{R}$ by

$$\Psi(\ell) = -\frac{1}{K+1} \sum_{\Gamma_i} \sum_{a \in H_1(\Gamma_i, \bigcup \widetilde{\mathbb{B}_j}; \mathbb{Z}_2) \setminus \{0\}} \log \widehat{\ell_i}(\widehat{a}), \tag{3.4}$$

where $\widehat{\ell_i}$ is the lift of $\widehat{\ell}$ to $\widehat{\Gamma}_i$.

Proposition 86 ([2] Proposition 16)**.** If $\ell \in \Sigma_{\widehat{\Gamma}}$ and $\tau \in T_\ell(\Sigma_{\widehat{\Gamma}})$ is integrable then

$$\|(\ell, \tau)\|^N = \|(\ell, \tau)\|^L + d_\tau \Psi,$$

where $d_\tau \Psi$ is the derivative of Ψ in the direction of τ, i.e., the derivative from the right at 0 of $t \mapsto \Psi(\ell + t\tau)$.

We can easily extend this discussion to the whole modified relative outer space $\mathrm{CV}'_n(\mathcal{A})$. It is easy to see that $\|\cdot\|^L, \|\cdot\|^N$ and Ψ commute with inclusions of simplices corresponding to collapsing nontrivial forests. If $\phi : R'_{n,k}(\mathcal{A}) \longrightarrow \Gamma$ is a marking, $\phi_* : H_1(R'_{n,k}(\mathcal{A}), \bigcup \mathbb{B}_j; \mathbb{Z}_2) \longrightarrow H_1(\Gamma, \bigcup \mathbb{B}_j; \mathbb{Z}_2)$ is an isomorphism and we identify homology classes in $H_1(\Gamma, \bigcup \mathbb{B}_j; \mathbb{Z}_2)$ with homology classes in $H_1(R'_{n,k}(\mathcal{A}), \bigcup \mathbb{B}_j; \mathbb{Z}_2)$. Similarly, cohomology can be identified, i.e., the double covers of Γ with double covers of $R'_{n,k}(\mathcal{A})$ (relatively to the wedge cycles), and ϕ lifts to markings of double covers of Γ by double covers of $R'_{n,k}(\mathcal{A})$ (relatively to the wedge cycles). This means that $\Psi : \mathrm{CV}'_n(\mathcal{A}) \longrightarrow \mathbb{R}$ can be defined globally. Moreover, Ψ is $\mathrm{Out}(F_n; \mathcal{A})$-invariant. The following theorem is the analog of a theorem due to Handel-Mosher [28] for $\mathrm{Out}(F_n)$.

Theorem 87. For any irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$, let λ be the expansion factor of Φ, i.e., its Frobenius eigenvalue, and μ the expansion factor of Φ^{-1}. Then $\mu \leq \lambda^C$.

The proof of this theorem is similar to the proof of Theorem 23 in [2]. However, the key idea is to find the connection between this norm and the computation of the distance of two points in the modified relative outer space. This goal is achieved considering piecewise linear paths $\gamma : [0, 1] \longrightarrow \mathrm{CV}'_n(\mathcal{A})$ and integrating on them. See [24] or [29] for a definition of piecewise linear path in outer space. For the definition in the modified relative outer space, we can define γ constructing a folding path as in Chapter 5. We recall the following definition:

Definition 88. For $\theta > 0$, the *θ-thick part* of $\mathrm{CV}'_n(\mathcal{A})$ is

$$\mathrm{CV}'_n(\mathcal{A})^\theta = \{X \in \mathrm{CV}'_n(\mathcal{A}) \,|\, l(\widehat{\alpha}, X) \geq \theta \text{ for all } \alpha \subset X\}.$$

Recall that $\mathrm{CV}'_n(\mathcal{A})^\theta$ is a $\mathrm{Out}(F_n; \mathcal{A})$-invariant closed subset on which $\mathrm{Out}(F_n; \mathcal{A})$ acts cocompactly.

Theorem 89. For every $\theta > 0$ there is a constant D such that for any $X, Y \in \mathrm{CV}'_n(\mathcal{A})^\theta$

$$d(Y, X) \leq D\, d(X, Y).$$

The proof is a combination of the Main Theorem in [2] and Proposition 69.

Corollary 90. For any $\theta > 0$ there is constant $c = c(\theta)$ such that:

$$\frac{1}{c} \cdot d(Y, X) \leq d(X, Y) \leq c \cdot d(Y, X)$$

for any $X, Y \in \mathrm{CV}'_n(\mathcal{A})^\theta$.

3.6 Nielsen Paths, Geometric and Nongeometric Automorphisms

Let $f : \Gamma \to \Gamma$ be a train track map representing $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$. In the following we will consider paths not completely contained in the wedge cycles. A path σ in Γ is a *periodic Nielsen path* if $[f^p(\sigma)] \simeq \sigma$ relatively to the wedge cycles for some minimal p called the *period* of σ. A *Nielsen path* is a periodic Nielsen path with period 1. A periodic Nielsen path is *indivisible* if it cannot be written as nontrivial concatenation of periodic Nielsen subpaths. It is obvious that every Nielsen path (pulled tight) is a concatenation of indivisible Nielsen paths.

Definition 91. Let $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$ be fully irreducible, $f : \Gamma \to \Gamma$ a topological representative of Φ which is a train track map. We say that f is $\mathcal{A}$-*Nielsen minimized* if the following conditions hold:

1. Every indivisible periodic Nielsen path $\rho \subset \Gamma$ that is not contained in a wedge cycle has period one.
2. The number of indivisible Nielsen paths that is not contained in a wedge cycle is as small as possible subject to the previous condition.

The next lemma is a simplified version of Lemma 5.2.3 in [9].

Lemma 92. For any fully irreducible $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$, (passing to a power of Φ if necessary) there exists an $\mathcal{A}$-Nielsen minimized train track map $f : \Gamma \to \Gamma$.

Suppose that $f : \Gamma \to \Gamma$ is a train track map. If $\rho \subset \Gamma$ is an indivisible Nielsen path that is not contained in a wedge cycle, then $\rho = \alpha\beta$, where α and β are legal and the turn at the juncture of $\overline{\alpha}$ and β is illegal in $\widehat{\Gamma}$. The proof of the previous fact can be found in [9].

Proposition 93. If $f : \Gamma \to \Gamma$ is an $\mathcal{A}$-Nielsen minimized train track map, then there exists at most one indivisible Nielsen path $\rho \subset \Gamma$ that is not contained in a wedge cycle. Moreover, if there is such an indivisible Nielsen path ρ, then:

1. Its first and last (possibly partial) edges are contained in $\widehat{\Gamma}$.
2. The illegal turn of ρ in $\widehat{\Gamma}$ is the only illegal turn in $\widehat{\Gamma}$.
3. ρ crosses every edge in $\widehat{\Gamma}$ at least once.
4. Either ρ crosses every edge of $\widehat{\Gamma}$ exactly twice or ρ crosses some edge of $\widehat{\Gamma}$ exactly once.

The proof of Proposition 93 can be found in [9] (see Lemma 5.2.5).

Definition 94. We say that Φ is *geometric* if there exist indivisible Nielsen paths and they close up to yield periodic loops. Otherwise, Φ is called *nongeometric*.

3.7 Whitehead Outer Automorphisms

Nielsen was the first to provide a finite list of generators for $\mathrm{Out}(F_n)$ called the Nielsen generators. We describe a larger list of generators given by Whitehead.

Definition 95. Let $\{x_1, \ldots, x_n\}$ be a basis for F_n. Let $S \subset \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ and $a \in S$ so that $a^{-1} \notin S$. Then the Whitehead automorphism $\phi_{(S,a)}$ associated with (S, a) is defined as follows. $\phi_{(S,a)}(a) = a$ and for $x \neq a, a^{-1}$:

$$\begin{array}{ll} x \longrightarrow axa^{-1} & \text{if } x, x^{-1} \in S \\ x \longrightarrow xa^{-1} & \text{if } x \in S \text{ and } x^{-1} \notin S \\ x \longrightarrow ax & \text{if } x \notin S \text{ and } x^{-1} \in S \\ x \longrightarrow x & \text{if } x, x^{-1} \notin S. \end{array}$$

If ϕ is an automorphism we denote its equivalence class in $\mathrm{Out}(F_n)$ by $[\phi]$.

Theorem 96 (Nielsen, Whitehead)**.** The set

$$\{[\phi_{(S,a)}] \mid \text{ all possible } a, S\}$$

generates $\mathrm{Out}(F_n)$. In fact, the set of outer automorphisms associated to $S = \{a, b\}$ for all choices of a, b such that $a \neq b, b^{-1}$ is a generating set.

Corollary 97. The set

$$\{[\phi_{(S,a)}] \mid [\phi_{(S,a)}] \in \mathrm{Out}(F_n; \mathcal{A})\}$$

generates $\mathrm{Out}(F_n; \mathcal{A})$.

Proof. We will use definitions and notions in [35] Section 3.2. Consider $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$. We need to show that

$$\Phi = [\phi_{(S_1,a_1)}] \cdots [\phi_{(S_m,a_m)}]$$

for $[\phi_{(S_i,a_i)}] \in \mathrm{Out}(F_n; \mathcal{A})$. Note that if S does not contain any y_i^j, $[\phi_{(S,a)}] \in \mathrm{Out}(F_n; \mathcal{A})$. Therefore, we can suppose that $\phi \in \Phi$ is the identity on $x_1, \ldots, x_{n-\sum s(i)}$. Consider an element $y_i^j \in A_j$ such that $\phi : y_i^j \mapsto a\omega y_i^j \overline{\omega a}$, for $j \in \{1, \ldots, k\}$, $i \in \{1, \ldots, s(j)\}$, $a \in \mathcal{A} * B$, and ω is a word in $\mathcal{A} * B$ (maybe empty). Let $S = \{y_i^j, \overline{y}_i^j\}$. The element $\phi_{(S,a)} \in \mathrm{Aut}(F_n; \mathcal{A})$ and $\phi_{(S,a)}^{-1} \cdot \phi$ has total x-length less than ϕ. Proceeding as in [35], it is possible to run an inductive argument on the total x-length. Because the total x-length is strictly decreasing, after a finite number of steps,

$$\phi^{-1}_{(S_1,a_1)} \cdots \phi^{-1}_{(S_m,a_m)} \cdot \phi = \mathrm{id}$$

and this concludes the proof of the corollary. □

3.7.1 Relative Whitehead Graphs

We say that the set of conjugacy classes $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_l\}$ in $\mathcal{A} * B$ *can be completed to a basis* if there are $w_i \in \mathfrak{a}_i$ and $w_{l+1}, \ldots, w_n$ such that $\{w_1, \ldots, w_n\}$ is a basis for $\mathcal{A} * B$. A conjugacy class α is a *basis element* if $\{\alpha\}$ can be completed to a basis of $\mathcal{A} * B$. We say that a basis element α is *primitive* if α and the generators of $\mathcal{A}$ can be completed to a basis. Otherwise, α is called nonprimitive. Recall the definition of the standard Whitehead graph.

Definition 98. Let $\mathcal{B} = \{y_1, \ldots, y_n\}$ be a basis of F_n, let $\mathfrak{a}$ be a conjugacy class in F_n, and $w \in \mathfrak{a}$ a cyclically reduced word written in the basis $\mathcal{B}$. Then the *Whitehead graph* of $\mathfrak{a}$ with respect to $\mathcal{B}$ is denoted $\mathrm{Wh}_{\mathcal{B}}(\mathfrak{a})$ and constructed as follows:

- The vertex set of this graph is the set $\mathcal{B} \cup \mathcal{B}^{-1}$;
- z_i and z_j are connected by an edge if $z_i^{-1} z_j$ or $z_j^{-1} z_i$ appears in the cyclic word w, i.e., if $w = \ldots z_i^{-1} z_j \ldots$ or $w = \ldots z_j^{-1} z_i \ldots$ or $w = z_j \ldots z_i^{-1}$ or $w = z_i \ldots z_j^{-1}$.

The Whitehead graph $\mathrm{Wh}_{\mathcal{B}}([w_1], \ldots, [w_k])$ of the set $\{[w_1], \ldots, [w_k]\}$ is the union of all the individual Whitehead graphs, taken with the same vertex set:

$$\mathrm{Wh}_{\mathcal{B}}([w_1], \ldots, [w_m]) = \cup_{i=1}^{m} \mathrm{Wh}_{\mathcal{B}}([w_i]).$$

We give a definition of Whitehead graph relative to $\mathcal{A}$.

Definition 99. Let $\mathcal{B} = \{y_1^1, \ldots, y_{s(k)}^k, x_1, \ldots, x_{n-\sum s(i)}\}$ be a basis of $\mathcal{A} * B$ and $\mathcal{B}_1 = \{x_1, \ldots, x_{n-\sum s(i)}\}$. Let $\mathfrak{a}$ be a conjugacy class in $\mathcal{A} * B$, and $w \in \mathfrak{a}$ a cyclically reduced word written in the basis $\mathcal{B}$. We will denote by $\widehat{w}$ the word obtained by w sending $y_i^j \mapsto 1$, $\forall i, j$ but without performing any cancelations afterwards. Then the *Whitehead graph of* $\mathfrak{a}$ *with respect to* $\mathcal{B}$ *and relative to* $\mathcal{A}$ is denoted by $\mathrm{Wh}_{\mathcal{B},\mathcal{A}}(\mathfrak{a})$ and constructed in the following way:

- The vertex set of this graph is the set $\mathcal{B}_1 \cup \mathcal{B}_1^{-1}$;
- z_i and z_j are connected by an edge if $z_i^{-1} z_j$ or $z_j^{-1} z_i$ appears in the cyclic word $\widehat{w}$, i.e., if $\widehat{w} = \ldots z_i^{-1} z_j \ldots$ or $\widehat{w} = \ldots z_j^{-1} z_i \ldots$ or $\widehat{w} = z_j \ldots z_i^{-1}$ or $\widehat{w} = z_i \ldots z_j^{-1}$.

Notice that there might be loops in the Whitehead graph. For example, if $\mathcal{B} = \{y_1, x_1, x_2\}$ and $w = x_1^{-1}y_1x_1x_2$, then $\widehat{w} = x_1^{-1}y_1x_1x_2$ and we get a loop in $\mathrm{Wh}_{\mathcal{B}}([w])$ (see Figure 3.8).

Definition 100. A *cut vertex* in a graph is a vertex that if removed, leaves the graph disconnected.

Theorem 101 (Whitehead's Theorem)**.** If $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ can be completed to a basis and $\mathrm{Wh}_{\mathcal{B}}(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ is connected, then $\mathrm{Wh}_{\mathcal{B}}(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ has a cut vertex.

Whitehead's Theorem 101 was proved by Whitehead in [45] only for the case $k = 0$. However, the proof works also for the general case without modifications, so we omit it.

Theorem 102 ([36])**.** The following are equivalent:

1. α is a primitive basis element.

2. If $\mathcal{B}$ is a basis such that $\mathrm{Wh}_{\mathcal{B},\mathcal{A}}([\alpha])$ contains no cut vertex, then $\mathrm{Wh}_{\mathcal{B},\mathcal{A}}([\alpha])$ is disconnected.

We have defined the Whitehead graph of a conjugacy class α in the basis $\mathcal{B}$. If $R \in \mathrm{CV}'_n(\mathcal{A})$ is a relative rose, then its marking identifies its edges with a basis $\mathcal{B}(R)$ of $\mathcal{A} * B$. The Whitehead graph of α in R is $\mathrm{Wh}_R(\alpha) = \mathrm{Wh}_{\mathcal{B}(R)}(\alpha)$ and the Whitehead graph of α in R relative to $\mathcal{A}$ is $\mathrm{Wh}_{R,\mathcal{A}}(\alpha) = \mathrm{Wh}_{\mathcal{B}(R),\mathcal{A}}(\alpha)$.

3.8 Basis Elements

For $X = (\Gamma, \phi) \in \mathrm{CV}'_n(\mathcal{A})$, any conjugacy class α of F_n may be identified with an immersed loop α_X in X.

Proposition 103. Let α_X, β_X be different candidates in X. Then there is a primitive basis element γ so that $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ can each (separately) be completed to a basis of $\mathcal{A} * B$.

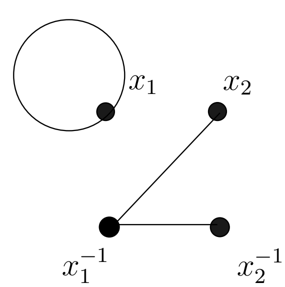

Figure 3.8. The Whitehead graph $Wh_{\mathcal{B}}([w])$, where $w = x_1^{-1}y_1x_1x_2$.

Proof. Suppose α_X is a candidate and γ_X is an embedded loop of positive length such that $\gamma_X \setminus \alpha_X \supseteq \{e_i\}$, $l(e_i, X) > 0$. Let J be a maximal forest in X which does not contain e_i. Collapse J to a rose R. Since e_i was not collapsed, $\gamma_R \setminus \alpha_R \supseteq \{e_i\}$. Let e_j be any edge that α crosses exactly once. Then $< \alpha_R, \gamma_R, e_1, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots e_n >$ represents a basis for $\mathcal{A} * B$. Suppose $\widehat{\alpha}$ is a figure 8, a barbell, or a loop of positive length and with an arc attached candidate and γ is an embedded circle so that $\gamma \subseteq \alpha$. Choose an edge e_i in $\alpha \setminus \gamma$ which α crosses only once. Collapse a maximal forest that does not contain e_i. Now choose e_j in γ_R. Then $< \alpha_R, \gamma_R, e_1, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots e_n >$ is a basis for $\mathcal{A} * B$. Let α_X, β_X be any two candidates. If one of them is an embedded loop whose image is not contained in the other, then α, β can be completed to a basis. If α, β have the same image, then find an embedded loop γ as in the previous paragraph so that α, γ and β, γ can be completed to a basis. If they have different images and are not embedded, let γ be an embedded loop so that $\mathrm{Im}\gamma \subseteq \mathrm{Im}\alpha$. Then α, γ can be completed to a basis. If $\mathrm{Im}\gamma \subseteq \mathrm{Im}\beta$, then β, γ can be completed to a basis. If γ is not contained in β, then again by the previous paragraph β, γ can be completed to a basis. A similar argument works also if $\widehat{\alpha}$ is an arc candidate. In any case we obtain a primitive basis element γ so that $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ can each be completed to a basis of $\mathcal{A} * B$. □

In this chapter we introduced the modified relative outer space on which $\mathrm{Out}(F_n; \mathcal{A})$ acts and we defined the train tracks for relative outer automorphisms, and the Lipschitz metric on this space. Moreover, we proved the Relative Train Track Theorem. Finally, we studied its tangent spaces, we proved that the Lipschitz metric is almost symmetric, and we introduced the background for the next chapters.

CHAPTER 4

DYNAMICS IN THE MODIFIED RELATIVE OUTER SPACE

The goal of this chapter is to study the dynamics in the modified relative outer space. First we define stable and unstable laminations of fully irreducible relative outer automorphisms. We give two equivalent definitions of laminations. For the first definition we follow the approach in [8] for outer space, while the second definition is a particular case of the notion of lamination in [9]. Then we study the stabilizer of a stable lamination proving that, modulo the kernel of the action, it is virtually cyclic. This fact was proved by Bestvina, Feighn, and Handel in [9] in a general setting. Here we give a different proof following [8]. Finally, we analyze the relationship between trees and laminations proving that an irreducible relative outer automorphism with irreducible powers acts on the modified relative outer space with north-south dynamics. This is the relative version of the theorem proved in [34] for $\mathrm{Out}(F_n)$.

4.1 Stable and Unstable Laminations

In this section we give two equivalent definitions of stable and unstable laminations associated to a fully irreducible relative outer automorphism. For the first definition we follow the approach in [8] for outer space, while the second definition is a particular case of the notion of lamination in [9]. We will use the same notation of the previous chapter.

4.1.1 First Definition of Lamination

Let $f : \Gamma \to \Gamma$ be a train track map with an irreducible relative transition matrix that is not periodic. *We will suppose that f is the identity on the wedge cycles and it has exponential growth.* We endow Γ with the structure of a modified marked metric $(\mathcal{A}, n)$-graph so that $\widehat{f}$ expands lengths of edges in $\widehat{\Gamma}$ by the expansion factor $\lambda > 1$. We can suppose that $\widehat{\Gamma}$ does not have edges with length 0. Let $x \in \Gamma$ be any f-periodic point in the interior of some edge not in a wedge cycle, that is there exists $N > 0$ such that $f^N(x) = x$. Let $\varepsilon > 0$ be small enough such that for the ε-neighborhood I of x in $\widehat{\Gamma}$ we have $\widehat{f}^N(I) \supset I$, where $\widehat{f} : \widehat{\Gamma} \to \widehat{\Gamma}$ is obtained by f collapsing the wedge cycles to (special) points. Choose an isometry $\ell : (-\varepsilon, \varepsilon) \to I$ and extend it to the unique locally isometric immersion $\ell : \mathbb{R} \to \widehat{\Gamma}$ such that $\ell(\lambda^N t) = \widehat{f}^N(\ell(t))$. We say that ℓ is obtained by iterating a neighborhood of $x \in \widehat{\Gamma}$.

Definition 104. Two isometric immersions $[a_1, b_1] \to \widehat{\Gamma}$ and $[a_2, b_2] \to \widehat{\Gamma}$ are *equivalent* if there is an isometry $[a_1, b_1] \to [a_2, b_2]$ such that the diagram

$$\begin{tikzcd} [a_1, b_1] \arrow[r] & \widehat{\Gamma} \\ [a_2, b_2] \arrow[u] \arrow[ur] & \end{tikzcd}$$

commutes.

If $\mathfrak{g}$ is an equivalence class represented by $\gamma : [a, b] \to \widehat{\Gamma}$, define $f(\mathfrak{g})$ to be the equivalence class of $[\widehat{f} \circ \gamma]$ scaled so that it is an isometric immersion.

A *leaf segment* of an isometric immersion $\mathbb{R} \to \widehat{\Gamma}$ is the equivalence class of the restriction to a finite interval. Two isometric immersions $\ell, \ell' : \mathbb{R} \to \widehat{\Gamma}$ are *(weakly) equivalent* if every leaf of ℓ is also a leaf of ℓ' and if every leaf of ℓ' is also a leaf of ℓ. The collection of leaf segments does not depend on the choice of x and I: if $x' \in \Gamma$ is another f-periodic point in the interior of some edge of Γ not in any wedge cycle, and $\ell' : \mathbb{R} \to \widehat{\Gamma}$ is obtained by iterating a neighborhood of x', then ℓ and ℓ' are equivalent. The proof of this fact is similar to the proof of Lemma 1.2 in [8], so we omit it.

Definition 105. The *stable lamination* $\Lambda = \Lambda_f^+(\Gamma)$ associated to $f : \Gamma \to \Gamma$ is the equivalence class of isometric immersions containing some immersion obtained by iterating a neighborhood of a periodic point as above. A *leaf* of Λ is any immersion representing Λ. A *stable leaf segment of* Λ is a leaf segment of some leaf ℓ of Λ.

Remark 106.

1. Any $\widehat{f}$-iterate of a leaf segment is a leaf segment.
2. Any edge of $\widehat{\Gamma}$ is a leaf segment of Λ.
3. Any subsegment of a leaf segment is a leaf segment. Any leaf segment is a subsegment of a sufficiently high iterate of an edge.
4. For any leaf segment $\mathfrak{g}_1$ there is a leaf segment $\mathfrak{g}_2$ such that $\mathfrak{g}_1 = f(\mathfrak{g}_2)$.
5. If α is an immersed loop in Γ and k is the number of edges $\widehat{\alpha}$ crosses in $\widehat{\Gamma}$ counted with multiplicities, then any $\widehat{[f^i(\alpha)]}$ can be written as a concatenation of $\leq k$ leaf segments.

Definition 107. We say that a sequence $\{\alpha_i\}$ obtained from isometric immersions $\alpha_i : S_i^1 \to \widehat{\Gamma}$, and the metric on S_i^1 is a scalar multiple of the standard path metric and the scalar depends on i, *weakly converges to* Λ if for every $L > 0$

$$\frac{m(\{x \in S_i^1 | \text{ the } L\text{-neighborhood of } x \text{ is a leaf segment}\})}{m(S_i^1)} \xrightarrow[i\to\infty]{} 1,$$

where m is the scaled Lebesgue measure.

As a consequence of Remark 106 we have the following proposition.

Proposition 108. Suppose that α is an immersed loop in Γ not fully contained in the wedge cycles and representing a f-nonperiodic conjugacy class. Then the sequence $\widehat{[f^i(\alpha)]}$ weakly converges to Λ.

Definition 109. An isometric immersion $\ell : \mathbb{R} \to \widehat{\Gamma}$ is *quasi-periodic* if for every $L > 0$ there exists $L' > L$ such that every leaf segment of ℓ of length L occurs as a subleaf segment of any leaf segment of ℓ of length L'.

In other words, as remarked in [1], we can think of ℓ as a necklace made of beads, and the segments of length L that appear in ℓ are beads with different colors. Then ℓ is quasi-periodic if any subchain of $\frac{L}{L'}$ consecutive beads, we can find beads of all possible colors. Note that in the relative case the beads are in $\widehat{\Gamma}$ (not in Γ).

Proposition 110. Any leaf of $\Lambda_f^+(\Gamma)$ is quasi-periodic.

See [8] for a proof of this proposition. Suppose that $f : \Gamma \to \Gamma$ and $\Lambda = \Lambda_f^+(\Gamma)$ are as above, and let Γ_1 be any other modified marked $(\mathcal{A}, n)$-graph. By $h : \Gamma \to \Gamma_1$ denote the homotopy equivalence relative to the wedge cycles corresponding to the difference of the markings. For any isometric immersion $\ell : \mathbb{R} \to \widehat{\Gamma}$ denote by $h(\ell)$ the unique (up to precomposition by an isometry of $\mathbb{R}$) isometric immersion $\mathbb{R} \to \widehat{\Gamma}_1$ proper homotopy equivalent to $\widehat{h} \circ \widehat{f}(\ell)$.

Lemma 111. We have the following:

1. If $\ell, \ell' : \mathbb{R} \to \widehat{\Gamma}$ are equivalent, then $h(\ell)$, $h(\ell')$ are equivalent.

2. If ℓ is quasi-periodic, then $h(\ell)$ is quasi-periodic.

Proof. The proof is a consequence of the fact that h can be factored as the composition of a homeomorphism which fixes the wedge cycles and changes the metric on each edge that is not in a wedge cycle, and a finite sequence of folds (which are defined in Section 4.1).

Essentially the proof of this fact can be found in [41]. However, we require that each map in the factorization sends each wedge cycle to itself. Since the statement is obvious for homeomorphisms and folds, the lemma follows. □

Definition 112. The *stable lamination of* $f : \Gamma \to \Gamma$ *in the* Γ_1*-coordinates* is the equivalence class $\Lambda^+_f(\Gamma_1)$ containing $h(\ell)$ for some leaf ℓ of $\Lambda^+_f(\Gamma)$.

A a consequence of Proposition 108 we have the following result.

Lemma 113. Let α be a loop in Γ representing a f-nonperiodic conjugacy class. Then the sequence $\{[\widehat{h(f^i(\alpha))}]\}$ weakly converges to $\Lambda^+_f(\Gamma_1)$.

The next lemma shows that Λ does not depend on the representative of the relative outer automorphism Φ.

Lemma 114. Suppose $g : \Gamma_1 \to \Gamma_1$ is a train track map representing the same fully irreducible relative outer automorphism Φ as $f : \Gamma \to \Gamma$. Then $\Lambda^+_f(\Gamma_1) = \Lambda^+_g(\Gamma_1)$.

Proof. By definition, g represents the same fully irreducible relative outer automorphism Φ as f if there exists a difference of the markings $h : \Gamma \to \Gamma_1$ such that $h \circ f \simeq g \circ h$ rel. $\bigcup_{j=1}^k \mathbb{B}_j$. Let α be a loop in Γ representing a f-nonperiodic conjugacy class. The sequence $\{[\widehat{h(f^i(\alpha))}]\} = \{[\widehat{g^i(h(\alpha))}]\}$ weakly converges to $\Lambda^+_g(\Gamma_1)$ by Proposition 108, and to $\Lambda^+_f(\Gamma_1)$ by Lemma 113. Therefore, $\Lambda^+_f(\Gamma_1)$ and $\Lambda^+_g(\Gamma_1)$ have arbitrarily long common leaf segments. Because they are both quasi-periodic, $\Lambda^+_f(\Gamma_1) = \Lambda^+_g(\Gamma_1)$. □

Definition 115. The *stable lamination* Λ^+_Φ of a fully irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$ is the collection

$$\{\Lambda^+_f(\Gamma) \,|\, \Gamma \text{ is a modified } (\mathcal{A}, n)\text{-graph}\},$$

where $f : \Gamma \to \Gamma$ is a train track representative of Φ. The *unstable lamination* Λ^-_Φ of Φ is the stable lamination of Φ^{-1} and its leaf segments are called *unstable.*

We denote by $\mathcal{IL} = \mathcal{IL}(\mathcal{A}, n)$ the set of stable laminations Λ^+_Φ as Φ ranges over all fully irreducible relative outer automorphisms in $\mathrm{Out}(F_n; \mathcal{A})$. The group $\mathrm{Out}(F_n; \mathcal{A})$ acts on $\mathcal{IL}$ in the following way: for any $\Psi \in \mathrm{Out}(F_n; \mathcal{A})$, if ℓ is a leaf of Λ^+_Φ in the Γ coordinates and $f : \Gamma \to \Gamma$ is a homotopy equivalence representing Ψ, then $[\widehat{f(\ell)}]$ represents a leaf of $\Lambda^+_{\Psi\Phi\Psi^{-1}}$. Note that the infinite cyclic subgroup generated by Φ fixes Λ^+_Φ.

4.1.2 Second Definition of Lamination

We give another definition of stable (unstable) lamination for a fully irreducible relative outer automorphism Φ. Let $\mathcal{B}(\Gamma)$ denote the space of lines in the graph Γ.

Definition 116. We say that $\beta' \in \mathcal{B}(\Gamma)$ is *weakly attracted to* $\beta \in \mathcal{B}(\Gamma)$ *under the action of* Φ if $[\Phi^k(\beta')] \to \beta$. A subset $U \subset \mathcal{B}(\Gamma)$ is an *attracting neighborhood of* β *for the action of* Φ if $\Phi(U) \subset U$ and if $\{\Phi^k(U) \,|\, k \geq 0\}$ is a neighborhood basis for β in $\mathcal{B}(\Gamma)$.

Definition 117. A bi-infinite path σ in a marked graph Γ is *birecurrent* if every finite subpath of σ occurs infinitely often as an unoriented subpath of each end of σ. A line is birecurrent in Γ if the path representing it (with either choices of orientation) is birecurrent.

Definition 118. A closed subset Λ^+ of $\mathcal{B}(\Gamma)$ is a *stable lamination* for Φ if it is the closure of a single point β such that

1. It is birecurrent.

2. It has an attracting neighborhood for the action of some iterate of Φ.

3. It is not carried by a Φ-periodic free factor of rank one.

Note that this definition of stable lamination is more general than the first one. Indeed, Φ could be any relative outer automorphism (see [9]). However, we are interested only in the fully irreducible case.

The next lemma shows that the definition of stable lamination given in Definition 115 is equivalent to the definition given in Definition 118.

Lemma 119. The definitions of stable lamination given in Definition 115 and Definition 118 are equivalent.

Before we prove Lemma 119, we need to understand better the properties of the first definition of lamination. Let $f : \Gamma \to \Gamma$ be a train track realization of a fully irreducible relative outer automorphism Φ. We equip $\widehat{\Gamma}$ with a minimal train track structure: we declare a turn legal if and only if it is crossed by leaves of $\Lambda^+_\Phi(\Gamma)$. Notice that the relative transition matrix of f is irreducible. Let the *local Whitehead graph* at a vertex v of $\widehat{\Gamma}$ be the graph formed by $\mathrm{Lk}(v, \widehat{\Gamma})$ by connecting two points if and only if the corresponding turn is legal.

Lemma 120. Suppose $\pi : \Gamma' \to \Gamma$ is a finite sheeted covering space and $f' : \Gamma' \to \Gamma'$ a lift of f. Then f' satisfies

1. the relative transition matrix of f' is irreducible. In particular, the legal structure on the turns in $\widehat{\Gamma}'$ and the local Whitehead graph of f' at a vertex v of $\widehat{\Gamma}'$ are well defined,

2. the local Whitehead graph of f' at a vertex v of $\widehat{\Gamma}'$ is the lift of the local Whitehead graph of f at $\pi(v)$, and in particular, it is connected.

The proof of this lemma is essentially the same as the proof of Lemma 2.1 in [8] if we replace G and g' by $\widehat{\Gamma}$ and $\widehat{f}'$.

Definition 121. Consider $F_n = \mathcal{A} * B$. We say that the conjugacy class of a subgroup $F < \mathcal{A} * B$ *carries* Λ^+_Φ if there is a modified marked $(\mathcal{A}, n)$-graph Γ, an isometric immersion $\iota : G \to \Gamma$ of graphs with $\mathrm{Im}(\pi_1(\iota)) = F \not< \mathcal{A}$, and an isometric immersion $\ell : \mathbb{R} \to \widehat{G}$ such that $\widehat{\iota} \circ \ell$ is a leaf of $\Lambda^+_\Phi(\Gamma)$, where $\widehat{G}$ is obtained from G by collapsing the preimage of the wedge cycles.

Remark 122. Notice that the definition does not depend on ℓ and Γ. Indeed, if one leaf lifts to $\widehat{G}$, so does every leaf. Moreover, if we find a factorization as above using a modified marked $(\mathcal{A}, n)$-graph Γ, then we can find a similar factorization using any modified marked $(\mathcal{A}, n)$-graph.

Proposition 123. If a finitely generated subgroup $F < \mathcal{A} * B$ carries Λ^+_Φ, then F has finite index in $\mathcal{A} * B$.

Proof. Let $f : \Gamma \to \Gamma$ be a train track representative of a fully irreducible relative outer automorphism Φ, and let $G \to \Gamma$ be an isometric immersion corresponding to F. Then G is a finite graph. Complete the immersion by adding vertices and edges to G to a connected finite-sheeted covering space $\Gamma' \to \Gamma$ (see [41]). Notice that if F has infinite index, then we really are adding new edges, and hence, a lift of a leaf does not cross every edge of $\widehat{\Gamma}'$. Now, pass to a further finite cover to which f lifts. But the leaves of the stable lamination of the lift do not cross those edges that cover added edges, contradicting Lemma 120. $\square$

Remark 124. Λ cannot be carried by a proper free factor of $(\mathcal{A} * B)/\mathcal{A}$ or by a Φ-periodic free factor of rank 1.

We may now prove Lemma 119.

Proof. First we prove that Definition 115 implies Definition 118. The proof of quasi-periodicity in Proposition 110 implies that β is birecurrent. By Proposition 108, β has an attracting neighborhood for the action of some iterate of Φ. Remark 124 implies that β cannot contain a free factor of rank 1.

Now we prove that Definition 118 implies Definition 115. Let $\Lambda^+ \in \mathcal{L}(\Phi)$ and $f : \Gamma \to \Gamma$ be a train track representative of Φ. Let Λ^+ is the closure of a single point β as in Definition 118. Since β is birecurrent in Γ, β is a quasi-periodic segment. Combining (1), (2) and (3) in Definition 118, it is easy to prove that β contains a leaf segment and hence $\Lambda^+ = \Lambda^+_f(\Gamma)$. □

Notice that the definition of stable lamination in Definition 118 is coordinate free.

Remark 125. If we lift the leaf segments to the $(\mathcal{A} * B)$-trees with special points defined in Chapter 3, we can give a third definition of lamination as the set of $(\mathcal{A} * B)$-invariant unordered pairs of distinct elements of $\partial(\mathcal{A} * B)/\partial\mathcal{A}$.

4.1.3 Stabilizer of a Lamination

We define the stabilizer of a (stable) lamination and we prove that the stabilizer modulo the kernel of the action KA is virtually cyclic. Fix a fully irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$, a train track representative $g : \Gamma \to \Gamma$, and let $\Lambda = \Lambda^+_\Phi$. Define

$$\mathrm{Stab}(\Lambda) = \{\Psi \in \mathrm{Out}(F_n; \mathcal{A}) |\, \Psi(\Lambda) = \Lambda\}.$$

Let's give another characterization of train track maps. A *filtration* for a topological representation $\psi : \Gamma \to \Gamma$ is an increasing sequence of (not necessarily connected) invariant subgraphs

$$\Gamma_0 \subset \cdots \subset \Gamma_m = \Gamma,$$

where $\Gamma_0 = \bigcup_{j=1}^k \mathbb{B}_j$.

The subcomplex $\mathrm{cl}(\Gamma_i \setminus \Gamma_{i-1})$ is denoted by H_i and it is called the *ith stratum*. A turn with an edge in H_i and an edge in Γ_{i-1} is called *mixed turn* in (Γ_i, Γ_{i-1}). The edges of the *ith stratum* that are not in the wedge cycles determine a square matrix M_i called the transition submatrix for H_i. If M_i is irreducible, then it has Perron-Frobenius eigenvalue λ_i. We say that H_i is *exponentially growing* if $\lambda_i > 1$. Otherwise, we say that H_i is a *non-exponentially-growing* stratum.

Remark 126. If the filtration associated to an irreducible ψ is $\Gamma_0 \subset \Gamma_1 = \Gamma$, where Γ_0 is the set of wedge cycles, and for each legal path in $\beta \subset H_1$, $\psi(\beta)$ is a path that does not contain any illegal turn in H_1, then ψ is a train track map.

As in the case of reducible outer automorphisms in $\mathrm{Out}(F_n)$, it is possible to construct a relative train track map for the reducible relative outer automorphisms in $\mathrm{Out}(F_n;\mathcal{A})$.

Definition 127. We say that $f:\Gamma\to\Gamma$ is a *relative train track map* if for each exponentially growing stratum H_i the following statements hold:

1. $D\widehat{f}$ maps the set of oriented edges in H_i to itself (in particular, all mixed turns in $(\widehat{\Gamma}_i,\widehat{\Gamma_{i-1}})$ are legal);

2. If $\alpha\subset\Gamma_{i-1}$ is a nontrivial path with endpoints in $H_i\cap\Gamma_{i-1}$, then $[f(\alpha)]$ is a nontrivial sequence of paths with endpoints in $H_i\cap\Gamma_{i-1}$;

3. For each legal path $\beta\subset H_i$, $[f(\beta)]$ does not contain any illegal turn in H_i.

Theorem 128. For every relative outer automorphism $\Phi\in\mathrm{Out}(F_n;\mathcal{A})$ there exists a relative train track map $f:\Gamma\to\Gamma$ representing Φ.

Theorem 128 is a special case of Theorem 5.12 in [13] so we omit the proof. Now let's go back to the stabilizer $\mathrm{Stab}(\Lambda)<\mathrm{Out}(F_n;\mathcal{A})$. Note that $\mathrm{KA}<\mathrm{Stab}(\Lambda)$.

Proposition 129. Let $\Psi\in\mathrm{Stab}(\Lambda)$, and let $f:\Gamma\to\Gamma$ be a relative train track representative of Ψ. Then

1. $\widehat{f}$ has finite order on every proper $\widehat{f}$-invariant subgraph of $\widehat{\Gamma}$ without valence 1 vertices that is a union of strata, and

2. if f has no exponentially growing strata, then $\widehat{f}$ has finite order.

Proof. 1. If $\Psi\in\mathrm{KA}$, the statement is obvious. Hence, suppose $\Psi\notin\mathrm{KA}$. Let $\ell:\mathbb{R}\to\widehat{\Gamma}$ be a leaf of Λ and let $\Gamma^0\subset\widehat{\Gamma}$ be a proper f-invariant subgraph. By Proposition 123, ℓ is a concatenation of nondegenerate segments in Γ^0 and in $\Gamma\setminus\Gamma^0$. Notice that all segments in Γ^0 are Nielsen otherwise f-iteration will produce arbitrarily long leaf segments contained in Γ^0 contradicting quasi-periodicity. Again by quasi-periodicity, there is an upper bound to the length of segments in both Γ^0 and $\widehat{\Gamma}\setminus\Gamma^0$, and hence only finitely many segments occur. Start with the disjoint union M of copies of the segments and the natural immersion $M\to\widehat{\Gamma}$. Identify two points of M if they are mapped to the same point of $\widehat{\Gamma}$. Then fold to get an immersion $\pi:M'\to\widehat{\Gamma}$. If we collapse the wedge cycles to special points, by Proposition 123, because ℓ lifts to M' by construction, π must be a finite-sheeted covering space. Therefore, a power of any loop in Γ^0 lifts to M'. It follows that this power is a concatenation of paths in Γ^0

each of which has finite $\widehat{f}$-order. Thus every loop in Γ^0 has finite $\widehat{f}$-order and hence $\widehat{f}_{|\Gamma^0}$ has finite order by the Relative Train Track Theorem (Theorem 66).

2. This is a consequence of Lemma 3.1.14 in [9].

□

The following lemma is the relative version of Proposition 3.3 in [9].

Lemma 130. Suppose $f : \Gamma \to \Gamma$ represents a relative outer automorphism Ψ. Then there is a positive number $\lambda = \lambda(f, \Lambda)$ such that for every $\varepsilon > 0$ there is $N > 0$ so that if ℓ is a leaf segment of Λ of length $> N$, then

$$\lambda - \varepsilon < \frac{l([\widehat{f}(\ell)], \Gamma)}{l(\ell, \Gamma)} < \lambda + \varepsilon.$$

We say that a leaf segment is a K-tile if it is of the form $\widehat{f}^K(e)$ for some $e \subset \widehat{\Gamma}$.

Proof. By Perron-Frobenius Theorem, long leaf segments of Λ cross edges of $\widehat{\Gamma}$ with frequencies close to those determined by the components of Perron-Frobenius eigenvector. Fix a large number K and note that even longer leaf segments are concatenation of K-tiles, each occurring with definite frequency.

If $N >> 0$, we can think of ℓ as being a concatenation of such leaf segments. Notice that we can discard the short segments at the ends of ℓ because their contribution is negligible. Thus we can suppose that ℓ is a concatenation of K-tiles τ_j^K in $\widehat{\Gamma}$. Let C denote the bounded cancelation constant for $h : \Gamma \to \Gamma$. Denote by l_j the length of τ_j^K, by l_j^h the length of $[\widehat{h}(\tau_j^K)]$, and by N_j the number of occurrences of τ_j^K in ℓ. Then

$$l(\ell, \Gamma) = \sum_j N_j l_j$$

and

$$\sum_j N_j(l_j^h - 2C) \leq l([\widehat{h}(\ell)], \Gamma) \leq \sum_j N_j l_j^h.$$

It follows that

$$\frac{\sum_j N_j l_j^h}{\sum_j N_j l_j} - 2C\frac{\sum N_j}{\sum_j N_j l_j} \leq \frac{l([\widehat{h}(\ell)], \Gamma)}{l(\ell, \Gamma)} \leq \frac{\sum_j N_j l_j^h}{\sum_j N_j l_j}.$$

Since $2C\frac{\sum N_j}{\sum_j N_j l_j} \to 0$ as $K \to \infty$ and the frequencies $\frac{N_j}{\sum_j N_j}$ converges to the coordinate of the Perron-Frobenius eigenvector as $l(\ell, \Gamma) \to \infty$. See the proof of Proposition 3.3 in [9] and Section 7 in [34] for more details. □

Note that KA $\triangleleft$ Stab(Λ). Therefore, we can consider Stab(Λ)/KA.

Define $\sigma : \mathrm{Stab}(\Lambda)/\mathrm{KA} \to \mathbb{R}^+$ by $\sigma([\Psi]) = \log(\lambda)$ for a representative $f : \Gamma \to \Gamma$ of Ψ, where λ is the positive number $\lambda(f, \Lambda)$ in Lemma 130. If $\Psi' \in [\Psi]$ and f' is a train track representative of Ψ', then $\lambda(f', \Lambda) = \lambda(f, \Lambda)$. Hence, σ is well defined.

If $[\Psi_1], [\Psi_2] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ and λ_1, λ_2 are the positive numbers in Lemma 130 of Ψ_1 and Ψ_2 respectively, it is clear that $\sigma([\Psi_1\Psi_2]) = \sigma([\Psi_1])\sigma([\Psi_2])$. We next study the kernel and the image of σ.

Lemma 131. If $\Psi \in \mathrm{Stab}(\Lambda)$ is exponentially growing and reducible, then $\sigma([\Psi])$ is bounded away from 1. In particular, there is no exponentially growing reducible $[\Psi] \in \mathrm{Ker}(\sigma)$.

Proof. By Proposition 129(1), we may assume that the relative train track map $f : \Gamma \to \Gamma$ representing Ψ has $\widehat{f}$ of finite order on a subgraph Γ^0 and $\widehat{f}$ has an exponentially growing block $\widehat{\Gamma} \setminus \Gamma^0$. It is easy to see that length of leaves grow exponentially at the rate equal to the expansion factor of the relative transition matrix of f. $\square$

Now we generalize Lemma 131.

Lemma 132. Let $f : \Gamma \to \Gamma$ be a train track map representing a fully irreducible relative outer automorphism Ψ. Then for every $C > 0$ there is a number $M > 0$ such that if γ is any path with positive length, then one of the following holds:

1. $[f^M(\gamma)]$ contains a legal segment of length $> C$;

2. $[f^M(\gamma)]$ has fewer illegal turns than γ.

3. γ is a concatenation $a \cdot b \cdot c$ of paths such that $[f^M(b)]$ is Nielsen and a and c have length $\leq 2C$ and at most one illegal turn.

Proof. Choose M bigger than the number of legal edge-path of length $\leq 2C$. Suppose that (1) and (2) are not satisfied by a path γ. Therefore iteration of γ amounts to iterating maximal legal subsegments of γ and canceling portions of adjacent ones (because (2) is not satisfied). Moreover, each maximal legal segment of γ, except possibly the ones that contain endpoints, must have two iterates that after cancelation yield the same segment (because (1) is not satisfied). In particular, each such segment contains a preperiodic point so that these points subdivide $\gamma = a \cdot b_1 \cdot b_2 \cdots b_m \cdot c$, and we have (3). $\square$

Lemma 133. Let Ψ be a nongeometric irreducible relative automorphism, let $f : \Gamma \to \Gamma$ and $f' : \Gamma' \to \Gamma'$ be $\mathcal{A}$-Nielsen minimized train track representatives of Ψ and Ψ^{-1} respectively, and let $h : \Gamma \to \Gamma'$ and $h' : \Gamma' \to \Gamma$ be Lipschitz homotopy equivalences corresponding to differences of markings. Then for any $C > 0$ there are constants $N_0 > 0$ and L_0 such that if ι is an isometric immersion into Γ of the real line, a segment or a circle of length $\geq L_0$, and if ι' is the isometric immersion obtained from $h \circ \iota$ by pulling tight, then one of the following holds:

1. $[f^{N_0}\iota]$ contains a legal segment of length $> C$.

2. $[f'^{N_0}\iota']$ contains a legal segment of length $> C$.

Proof. Because f and f' are $\mathcal{A}$-Nielsen minimized and Ψ is nongeometric, we have only one Nielsen path and the starting point and the endpoint do not coincide. Therefore, it is impossible to concatenate Nielsen paths in Γ or Γ'. Without loss of generality, we may assume that C is larger than the critical constants for f and f'. Let M_1, M_2 be the constants obtained by Lemma 132 to h, C and h', C respectively. Let $M = \max\{M_1, M_2\}$. We will fix a large integer $s = s(f, f', h, h', M)$ that we will define precisely below. Suppose now that (1) does not hold with $N_0 = sM$. We will apply Lemma 132 only to segments $\gamma \subset \iota$ such that $f^M(\partial\gamma) \subset [f^M\iota]$. Call such segments f^M-admissible. By our assumption, possibility (1) never occurs. If you further restrict to segments γ with > 3 illegal turns, then (3) cannot hold either. Thus for such segments (2) always holds. We can represent ι as a concatenation of such segments of uniformly bounded length, and the uniform bound does not depend on ι, but only on f, f', h, h', M. We denote the number of illegal turns in $*$ by $\mathrm{nit}(*)$.

Say p is an upper bound to the number of illegal turns in each segment. Fix c with $\frac{p-1}{p} < c < 1$. For long enough segments γ in ι the ratio $\frac{\mathrm{nit}([\mathrm{f}^{\mathrm{M}}(\gamma)])}{\mathrm{nit}(\gamma)} < c$.

Applying inductively the same construction to $f^{rM}(\iota)$, $r \in \mathbb{Z}$, for a given $s > 0$ and long enough segments $L \subset \iota$ (the length depends on s as well) $\frac{\mathrm{nit}([\mathrm{f}^{\mathrm{sM}}(\mathrm{L})])}{\mathrm{nit}(\mathrm{L})} < c^s$. Otherwise (1) holds with $N_0 = sM$. Since legal segments in the graph obtained by collapsing the wedge cycles to special points have length bounded above by C and below by the length of the shortest edge with positive length (with the exception of the two edges containing the endpoints), the length is comparable to the number of illegal turns. Therefore, if (1) fails,

$$\frac{l([f^{sM}(\gamma)], \Gamma)}{l(\gamma, \Gamma)} < \ \mathrm{const}(f, C)c^s.$$

Repeat the discussion with $[hf^{sM}\iota]$ in place of ι, and with f' in place of f. If (2) fails as well (with $N_0 = sM$), we get

$$\frac{l([h'^{sM}\tau f^{sM}(\gamma)], \Gamma')}{l(hf^{sM}(\gamma), \Gamma')} < \ \mathrm{const}(f', C)c^s.$$

Multiplying, observing that $f'^{sM} h h^{sM} \simeq h'$ rel. $\bigcup_{j=1}^{k} \mathbb{B}_j$ and the homotopy is bounded so that for long γ the ratio

$$\frac{l([h'^{sM} \tau h^{sM}(\gamma)], \Gamma')}{l([h'(\gamma)], \Gamma')}$$

is in the interval $[\frac{1}{2}, 2]$. Hence,

$$\frac{1}{2}\mathrm{Lip}^{-1}(\tau)\mathrm{Lip}^{-1}(\tau') \leq \frac{l([f^{sM}(\gamma)], \Gamma)}{l([hf^{sM}(\gamma)], \Gamma')} \frac{l([h'(\gamma)], \Gamma')}{l(\gamma, \Gamma)} < 2 \text{ const}(f) \text{ const}(f') c^{2s}.$$

For $s >> 0$ we have a contradiction. $\square$

Definition 134. We say that a sequence $\{\Lambda_i\}$ of irreducible laminations in $\mathcal{IL}$ *weakly converges* to $\Lambda \in \mathcal{IL}$ if every leaf segment of Λ is a leaf segment of Λ_i for all but finitely many i.

Proposition 135. Let $\Lambda = \Lambda_\Phi^+ \in \mathcal{IL}$ be an irreducible lamination, and let Ψ and Φ be fully irreducible relative outer automorphisms in $\mathrm{Out}(F_n; \mathcal{A})$. Then either the forward Ψ-iterates of Λ weakly converge to Λ_Ψ^+ or $\Lambda = \Lambda_\Psi^-$. In particular, if $[\Psi] \in \mathrm{Stab}(\Lambda)/\mathrm{KA}$, then $\Lambda = \Lambda_\Psi^\pm$.

Proof. First, consider the case that Ψ is nongeometric. We use the notation from Lemma 133. Let ℓ be a leaf of Λ in the Γ-coordinates. We apply Lemma 133 to $[\widehat{f}^K(\ell)]$ with $K > 0$ and C larger than the critical constants (defined in Chapter 3) of f and f'. If for some $K > 0$ (1) holds, then it follows from quasi-periodicity that the forward iterates weakly converge to Λ_Ψ^+. The remaining possibility is that $[\widehat{h}\widehat{f}^K(\ell)]$ contains a $\widehat{\Gamma}$-legal segment of length $> C$ for all $K > 0$. But this means that $[\widehat{h}(\ell)]$, which equals $[\widehat{f'}^K \widehat{h}\widehat{f}^K(\ell)]$ up to bounded error, contains an arbitrarily high $\widehat{f'}$-iterate of a legal segment, hence $\Lambda = \Lambda_f^-$ (again by quasi-periodicity). If Ψ is geometric, see [9]. $\square$

As a consequence of Proposition 135, Lemma 131 holds without assuming that Ψ is reducible. We are now ready to prove the main result of this section.

Theorem 136. For every fully irreducible Φ, $\mathrm{Stab}(\Lambda_\Phi^+)/\mathrm{KA}$ is virtually cyclic.

Proof. The map $\sigma : \mathrm{Stab}(\Lambda_\Phi^+)/\mathrm{KA} \to \mathbb{R}^+$ has finite kernel. Because the image is an infinite and discrete subset of $\mathbb{R}^+$, it is isomorphic to $\mathbb{Z}$. Therefore, $\mathrm{Stab}(\Lambda_\Phi^+)/\mathrm{KA}$ is virtually cyclic. $\square$

Another proof of Theorem 136 can be found in [9].

Corollary 137. If $[\Psi] \in \mathrm{Stab}(\Lambda_\Phi^+)/\mathrm{KA}$ has $\sigma([\Psi]) \neq 1$, then $[\Psi]$ and $[\Phi]$ have common nonzero powers.

Proof. If $[\Psi] \in \mathrm{Stab}(\Lambda_{\Phi}^{+})/\mathrm{KA}$ and $\sigma([\Psi]) \neq 1$, then $\sigma([\Phi^n]) = \sigma([\Psi^m])$ for finitely many pairs of integers n and m. Since the kernel of σ is finite, there exist integers r and s such that $[\Phi^{n+r}] = [\Psi^{m+s}]$. □

Proposition 138. Let Φ and Ψ be two fully irreducible relative outer automorphisms. The following statements are equivalent:

1. $\{\Lambda_{\Phi}^{\pm}\} = \{\Lambda_{\Psi}^{\pm}\}$.

2. $\{\Lambda_{\Phi}^{\pm}\} \cap \{\Lambda_{\Psi}^{\pm}\} \neq \emptyset$.

3. $[\Phi]$ and $[\Psi]$ have common nonzero powers.

Proof. Obviously, (3) $\Rightarrow$ (1) $\Rightarrow$ (2). By Corollary 137 we have (2) $\Rightarrow$ (3). □

4.2 Trees and Modified Relative Outer Space

First, we define the boundary of the modified relative outer space. Then we study the relationship between trees and laminations.

4.2.1 Boundary of the Modified Relative Outer Space

We recall the following definition due to Cohen and Lustig.

Definition 139. An action of F_n on an $\mathbb{R}$-tree is *very small* if

1. all edge stabilizers are cyclic,

2. $\mathrm{Fix}(g)$ is isometric to a subset of $\mathbb{R}$ for $1 \neq g \in F_n$ and

3. $\mathrm{Fix}(g) = \mathrm{Fix}(g^i)$ for all $i \geq 2$.

Cohen and Lustig [20] showed that a simplicial action is in $\overline{\mathrm{CV}_n}$ if and only if it is very small. Bestvina and Feighn [6] proved that this is actually true for all actions concluding that the closure of outer space is the set of very small actions of F_n on $\mathbb{R}$-trees. In the relative case the main issue is understanding on which $\mathbb{R}$-trees the group $\mathcal{A} * B$ is acting on and those trees are the $(\mathcal{A} * B)$-trees with special points defined in Section 3.1. Recall that those trees with special points are the simplicial F_n-trees with elliptic subgroups $A_1, \ldots, A_k$. Notice that $\mathrm{CV}'_n(\mathcal{A})$ embeds naturally in the space of actions of $\mathcal{A} * B$ on metric $\mathbb{R}$-trees with special points such that all edge stabilizers are cyclic. However, the length function topology of the embedding in the space of actions is not the simplicial topology of $\mathrm{CV}'_n(\mathcal{A})$ described in Section 3.1. For example, in the case $n = 2$, $F_2 =< a, b >$ and $A =< a >$, the modified relative outer space $\mathrm{CV}_2'(A)$ is the union of half-open 1-simplices attached to a point (see Example 40). The simplicial topology assigns to each 1-simplex the same length, while the length function topology assigns a decreasing length for the two sequences of trees middle-points of the 1-simplices corresponding to the graphs with marking induced by $a \mapsto a$, $b \mapsto a^N b$ as $N \to +\infty$ and as $N \to -\infty$ respectively. We can consider the closure $\overline{\mathrm{CV}_n'}(\mathcal{A})$ of the image of the embedding. The image consists of projective classes of actions on metric $\mathbb{R}$-trees with special points where

1. all edge stabilizers are cyclic,
2. $\mathrm{Fix}(g)$ is isometric to a subset of $\mathbb{R}$ for $1 \neq g \in F_n/\mathcal{A}$,
3. the A_i's are the only elliptic elements, and
4. $\mathrm{Fix}(g) = \mathrm{Fix}(g^i)$ for all $i \geq 2$.

The boundary of the modified relative outer space is

$$\partial \mathrm{CV}_n'(\mathcal{A}) = \overline{\mathrm{CV}_n'}(\mathcal{A}) \setminus \mathrm{CV}_n'(\mathcal{A}).$$

In the future we will denote by $\mathrm{cv}_n(\mathcal{A})$ the unprojectivized modified relative outer space equipped with the length function topology. Moreover, we will denote by $\overline{\mathrm{cv}_n}(\mathcal{A})$ the closed unprojectivized modified relative outer space.

4.2.2 Trees and Laminations

In this section we examine the relationship between laminations and trees. Let $\mathfrak{T}$ denote the space of trees representing elements of $\overline{\mathrm{CV}_n'}(\mathcal{A})$. All the trees appearing in this section will be limits of free simplicial trees.

Definition 140. The *bounded cancelation constant* of a $(\mathcal{A} * B)$-map $f : T_0 \to T$, denoted $\mathrm{BCC}(f)$, is the least upper bound of numbers B with the property that there exist points a, b, c in T_0 with $b \in [a, c]$ such that the distance between $f(b)$ and $[f(a), f(c)]$ is B.

We need the following two lemmas. The first lemma is a generalization of the Bounded Cancelation Lemma.

Lemma 141. Let $f : T_0 \to T$ be an $(\mathcal{A} * B)$-map from a simplicial tree T_0 with special points to a tree $T \in \mathfrak{T}$. Then

$$\mathrm{BCC}(f) \leq \mathrm{Lip}(f)\mathrm{vol}(T_0),$$

where $\mathrm{vol}(T_0)$ is the relative volume of $T_0/(\mathcal{A} * B)$.

The proof is the same as the proof of Lemma 3.1 in [8].

Lemma 142. 1. There is a continuous section $\Sigma : \overline{\mathrm{cv}_n}(\mathcal{A}) \to \mathfrak{T}$.

2. Let $T_0 \in \mathfrak{T}$ be a free and simplicial tree with special points. For every $T \in \Sigma(\overline{cv_n}(\mathcal{A}))$ there is a $(\mathcal{A} * B)$-map $f_{T_0,T} : T_0 \to T$ that varies continuously with T and so that the Lipschitz constants $\mathrm{Lip}(f_{T_0,T})$ are uniformly bounded.

The proof of this lemma is similar to the proof for outer space (see [40] and [44]) so we omit it. Now, let Λ be the stable lamination of some irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$. Denote by T_1 the tree with special points associated to Γ.

Definition 143. Let Γ be a modified $(\mathcal{A}, n)$-marked graph. We say that the length of Λ in a tree is 0 and we write $l_T(\Lambda) = 0$, if there is a constant $C = C(\Gamma, T)$ such that if $\ell : \mathbb{R} \to \widehat{\Gamma}$ is a leaf of Λ and if $\widetilde{\ell} : \mathbb{R} \to T_1$ is a lift of ℓ to the tree corresponding to Γ, then for every segment $L \subset \mathbb{R}$ the length of $[f_{T_1,T}(L)]$ is $\leq C$.

Notice that the definition depends only on the projective class of T and not on the choice of Γ or $f_{T_1,T}$. Moreover, if T is simplicial, then the length of Λ in T is not 0. Recall that $\mathrm{Out}(F_n; \mathcal{A})$ acts on $\mathfrak{T}$ on the right in the following way:

$$l_{T\cdot\Phi}(\gamma) = l_T(\Phi(\gamma)).$$

The action induces an action of $\mathrm{Out}(F_n; \mathcal{A})$ on $\overline{\mathrm{CV}'_n}(\mathcal{A})$ and on $\overline{\mathrm{cv}_n}(\mathcal{A})$.

Definition 144. The *stable tree* $T^+ = T^+_\Phi$ of Φ is defined to be the limit of the sequence $\{T_1 \cdot g^i\}$ for a relative train track map $g : \Gamma \to \Gamma$ for Φ, where T_1 is the tree with special points associated to Γ. If λ is the expansion factor of g, then $[T^+]$ is the projective class of the tree T^+ so that

$$l_{[T^+]}(\gamma) = \lim_{i\to\infty} \frac{l(g^i(\gamma), \Gamma)}{\lambda^i},$$

and in particular

$$l_{[T^+]}(g(\gamma)) = \lambda l_{[T^+](\gamma)}.$$

The *unstable tree* $T^- = T^-_\Phi$ is defined to be the stable tree of Φ^{-1}.

That these definitions do not depend on the choice of a relative train track follows from the next lemma.

Lemma 145. If the length of Λ^+_Φ in $T \in \overline{\mathrm{cv}_n}(\mathcal{A})$ is not 0, then T converges to T^+_Φ under iteration by Φ. Moreover, the convergence is uniform on compact sets $K \subset \mathfrak{T}$, where Λ^+_Φ has nonzero length.

Proof. Let $g : \Gamma \to \Gamma$ be a train track representative for Φ. If ℓ is a sufficiently long leaf segment of Λ^+_Φ in the Γ-coordinates, then its image in $[T] \in \mathfrak{T}$ is long. Let β be a legal loop in Γ with positive length. If α is a g-nonperiodic loop in Γ (not completely contained in the wedge cycles), then for large N both $g^N(\alpha)$ and $g^N(\beta)$ can be viewed, up to bounded error, as a concatenation of long segments that occur with approximately the same frequencies. Therefore, since $l_{[T]}$ is constant on conjugacy classes

$$\lim_{N\to\infty} \frac{l_{[T]}(\Phi^N(\alpha))}{l_{[T]}(\Phi^N(\beta))}$$

is independent of $[T] \in \Sigma(K)$. If α is periodic, the above limit is again independent of $[T] \in \Sigma(K)$ and equal to 0. The claim now follows by observing that we may assume that $T^+_\Phi \in K$. The details of this argument is similar to those of the proof of Lemma 130. $\square$

Lemma 146. Let $g : \Gamma \to \Gamma$ be a relative train track representative for Φ. Let Λ be a stable lamination (perhaps unrelated to Φ).

1. For $T \in \overline{\mathrm{cv}_n}(\mathcal{A})$, if there is a leaf segment of Λ in Γ-coordinates whose image in $\Sigma(T)$ under $f_{T_1,\Sigma(T)}$ pulled tight has length $> 2\mathrm{BCC}(f_{T_1,\Sigma(T)})$, then Λ does not have length 0 in T.

2. The set $\{T \in \overline{\mathrm{cv}_n}(\mathcal{A}) \,|\, l_T(\Lambda) = 0\}$ is closed.

3. $l_{T^+_\Phi}(\Lambda^-_\Phi) = 0$ and $l_{T^-_\Phi}(\Lambda^+_\Phi) = 0$.

4. If $\Lambda \neq \Lambda^-_\Phi$, then Λ does not have length 0 in T^+_Φ.

Proof.
1. This follows from quasi-periodicity of Λ. If ℓ is a leaf segment whose image in $\Sigma(T)$ has length $2\mathrm{BCC}(f_{T_1,\Sigma(T)})+\varepsilon$, then a leaf segment of Λ that contains N disjoint copies of ℓ maps a segment, pulled tight, in $\Sigma(T)$ of length $\geq 2\mathrm{BCC}(f_{T_1,\Sigma(T)}) + N\varepsilon$.

2. This is a consequence of (1). If $T' \in \overline{\mathrm{cv}_n}(\mathcal{A})$ is such that $l_{T'}(\Lambda) \neq 0$, then there is a leaf segment ℓ in a leaf of Λ that maps to a segment in $\Sigma(T')$ of length greater than $2\mathrm{BCC}(f_{T_1,\Sigma(T')})$ for all $T_1 \in \mathfrak{T}$. In particular, this is true for all T in a neighborhood of T'.

3. Let γ be a loop in Γ representing a Φ-nonperiodic element. Let l be the length of γ in $\Sigma(T^-_\Phi)$. Then the length of $\Phi^N(\gamma)$ in $\Sigma(T^-_\Phi)$ is $l\lambda^{-N}$, and $l\lambda^{-N} \to 0$ as $N \to \infty$. But for large N the loop $\Phi^N(\gamma)$ will contain long leaf segments of Λ^+_Φ. These leaf segments map in $\Sigma(T^-_\Phi)$ to segments of length $\leq 2BCC(f_{T_1,\Sigma(T)})$ by an argument as in (1).

4. If Λ has length 0 in T^+_Φ, then all iterates of Λ by Φ have length 0 in T^+_Φ. It follows that $\Phi^N(\Lambda) \not\rightharpoonup \Lambda^+_\Phi$, otherwise some positive iterate would contain a leaf segment that maps to a long segment in $\Sigma(T^+_\Phi)$. By Proposition 135, $\Lambda = \Lambda^-_\Phi$.

□

Definition 147. A tree is *irreducible* if it is of the form T^+_Φ for some fully irreducible relative outer automorphism Φ.

Denote by $\mathcal{IT} = \mathcal{IT}(\mathcal{A}, n)$ the set of irreducible trees.

Definition 148. Two fully irreducible relative outer automorphisms Φ and Ψ are called *independent* if $\{T^\pm_\Phi\} \cap \{T^\pm_\Psi\} = \emptyset$.

Let $F : \mathcal{IT} \to \mathcal{IL}$ be the function that takes an irreducible tree $T = T^+_\Phi$ to the unique irreducible lamination Λ^+_Φ that has length 0 in T.

Corollary 149. The function $F : \mathcal{IT} \to \mathcal{IL}$ is a $\mathrm{Out}(F_n; \mathcal{A})$-equivariant bijection. In particular, $\mathrm{Stab}(T) = \mathrm{Stab}(F(T))$.

Proof. Obviously, F is $\mathrm{Out}(F_n; \mathcal{A})$-equivariant and surjective. If $F(T^+_\Phi) = F(T^+_\Psi)$, then both Φ and Ψ stabilize the same irreducible lamination, and hence have common powers. It follows that $T^+_\Phi = T^+_\Psi$, and so F is injective. □

In the next proposition we are going to use the following lemma.

Lemma 150 (Ping-pong Lemma)**.** Let G be a group acting on a set X. Let $a_1, \dots, a_k$ be elements of G. Suppose there exist disjoint nonempty subsets $X^+_1, \dots, X^+_k$ and $X^-_1, \dots, X^-_k$ of X with the following properties:

- $(X \setminus X^-_i) \cdot a_i \subset X^+_i$, for $i = 1, \dots, k$;
- $(X \setminus X^+_i) \cdot a_i^{-1} \subset X^-_i$, for $i = 1, \dots, k$.

Then the subgroup $H =< a_1, \dots, a_k >< G$ generated by $a_1, \dots, a_k$ is free with free basis $\{a_1, \dots, a_k\}$.

Proposition 151. Suppose that Φ and Ψ are two fully irreducible relative outer automorphisms such that $[\Phi], [\Psi] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ do not have common powers. Then there is $N \geq 1$ such that for any $m, r \geq N$, the subgroup $< [\Phi^m], [\Psi^r] >$ of $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ is free of rank two with free basis $[\Phi^m], [\Psi^r]$.

Proof. By Proposition 138 and Corollary 149, Φ and Ψ are independent, and the four laminations $\Lambda^{\pm}_{\Phi}$ and $\Lambda^{\pm}_{\Psi}$ are all distinct. By Lemma 146(4), the only lamination from these four that has length 0 in T^{+}_{Φ} is Λ^{-}_{Φ}. Similarly, an analogue statement is true for T^{-}_{Φ} and $T^{\pm}_{\Psi}$. By Lemma 146(2), there are pairwise disjoint compact neighborhoods $U^{\pm}_{\Phi}$ and $U^{\pm}_{\Psi}$ of $T^{\pm}_{\Phi}$ and $T^{\pm}_{\Psi}$ respectively so that Λ^{+}_{Φ} has nonzero length in every tree in U^{+}_{Φ} and $U^{\pm}_{\Psi}$. Similarly, an analogue statement is true for T^{-}_{Φ} and $T^{\pm}_{\Psi}$. By Lemma 145, there is $N > 0$ so that if $m, r \geq N$ then

- $(\overline{\mathrm{CV}'_n} \setminus U^{-}_{\Phi}) \cdot [\Phi^m] \subseteq U^{+}_{\Phi}$;
- $(\overline{\mathrm{CV}'_n} \setminus U^{+}_{\Phi}) \cdot [\Phi^{-m}] \subseteq U^{-}_{\Phi}$;
- $(\overline{\mathrm{CV}'_n} \setminus U^{-}_{\Psi}) \cdot [\Psi^r] \subseteq U^{+}_{\Psi}$;
- $(\overline{\mathrm{CV}'_n} \setminus U^{+}_{\Psi}) \cdot [\Psi^{-r}] \subseteq U^{-}_{\Psi}$.

By the Ping-pong Lemma, for every $m, r \geq N$, $[\Phi^m]$ and $[\Psi^r]$ freely generate F_2. $\square$

4.3 North-South Dynamics

The goal of this section is the proof of the following result.

Theorem 152. Each fully irreducible and exponentially growing relative outer automorphism Φ acts on the space $\overline{\mathrm{CV}'_n}(\mathcal{A})$ with north-south dynamics: $T^{\pm}_{\Phi}$ are the only fixed points, any compact set that does not contain T^{-}_{Φ} converges uniformly under iteration by Φ to T^{+}_{Φ}, and any compact set that does not contain T^{+}_{Φ} converges uniformly under iteration by Φ^{-1} to T^{-}_{Φ}.

This theorem was proved by Levitt and Lustig [34] in the case of CV_n. We will follow their approach proving the relative version of the theorem. If the proof of a result is similar to the proof of the analog result in [34], we omit the proof but instead we write the reference.

We will always assume that the actions on trees are in $\overline{\mathrm{cv}_n}(\mathcal{A})$. We will denote by $\overline{T}$ the metric completion of the tree T and by $(\mathcal{A} * B)$-tree the metric $\mathbb{R}$-tree with special points defined in Section 3.1. Let $f : T_0 \to T$ be a $(\mathcal{A} * B)$-equivariant map. A segment $[x, y]$ in T_0 is f-backtracking if $f(x) = f(y)$ and f has the *backtracking property* (BBT) if there exists a non-negative constant C such that the f-image of any segment $[x, y]$ in T_0 is contained in the C-neighborhood of $[f(x), f(y)]$. The smallest C with this property is called the BBT constant of f and it is denoted by $\mathrm{BBT}(f)$. Notice that $\mathrm{BBT}(f) = \mathrm{BCC}(f)$. By Lemma 141, $\mathrm{BBT}(f) \leq \mathrm{Lip}(f)\mathrm{vol}(T_0)$.

Proposition 153 ([34] Proposition 2.2)**.** Let T be a minimal $(\mathcal{A}*B)$-tree with dense orbits and trivial arc stabilizers. Given $\varepsilon > 0$, there exists a free simplicial $(\mathcal{A}*B)$-tree T_0 with $\mathrm{vol}(T_0) < \varepsilon$, and an equivariant map $f : T_0 \to T$ whose restriction to each edge is isometric.

As a consequence of Proposition 153 and Lemma 141 we have the following corollaries.

Corollary 154. Let T be a minimal $(\mathcal{A}*B)$-tree with dense orbits and trivial arc stabilizers. Given $\varepsilon > 0$, there exists a simplicial $(\mathcal{A}*B)$-tree T_0 with $\mathrm{vol}(T_0) < \varepsilon$, and an equivariant map $f : T_0 \to T$ with $\mathrm{BBT}(f) < \varepsilon$.

Corollary 155. Let T be a minimal $(\mathcal{A}*B)$-tree with dense orbits and trivial arc stabilizers. Given $P \in T$ and $\varepsilon > 0$, there exists a basis $\{x_1, \dots, x_{n-\sum_{i=1}^{k} s(i)}\}$ of $F_n/\mathcal{A}$ such that

$$\sum d(P, x_l P) < \varepsilon.$$

Another result following from Proposition 153 and Corollary 155 is the following.

Corollary 156. Given $P \in \overline{T}$ and $\varepsilon > 0$, there exists a simplicial $(\mathcal{A}*B)$-tree T_0 and $f : T_0 \to \overline{T}$ with $\mathrm{BBT}(f) < \varepsilon$ sending the vertex g to gP.

4.3.1 The point $Q(X)$

Given an $\mathbb{R}$-tree T, we define ∂T, the boundary of T, as the set of equivalence classes of rays $\rho : [0, \infty) \to T$, where ρ is an isometric map and $\rho_1 \sim \rho_2$ if the set $\{d(\rho_1(t), \rho_2(t)) \,|\, t \in [0, \infty)\}$ is bounded. We will always confuse a ray and its image. Let T and T_0 be as in Proposition 153 and $X \in \partial T_0$. Consider $f : T_0 \to T$ as before. We represent X by a ray ρ in T_0 and we consider $r = f(\rho)$.

Definition 157. We say that X is T-*bounded* if r is bounded in T.

Proposition 158. Let T be a minimal $(\mathcal{A}*B)$-tree with dense orbits and trivial arc stabilizers. If $X \in \partial T_0$, we can associate a unique point $Q(X)$ in $\partial T \sqcup \overline{T}$ and $Q : \partial T_0 \to \overline{T} \sqcup \partial T$ is $(\mathcal{A}*B)$-equivariant. Moreover, the map Q restricts to a bijection from the set of T-unbounded points of ∂T_0 onto ∂T.

This proposition is a consequence of the relative version of Proposition 3.1 in [34].

Lemma 159 ([34] Lemma 3.4)**.** Let $X \in \partial T_0$ be T-bounded and let $x \in T$ be a base point. For any $P \in \overline{T}$, there exists a sequence $\{g_n \in (\mathcal{A}*B)\}$ such that $g_n x \to X$ and $g_n P \to Q(X)$. Conversely, if $h_n x \to X$ and $h_n P$ converges to some $R \in \overline{T}$, then $R = Q(X)$.

Corollary 160. The map $Q : \partial T_0 \to \overline{T} \sqcup \partial T$ is onto.

Denote by $\mathfrak{T}'$ the space of $(\mathcal{A} * B)$-trees with dense orbits and trivial arc stabilizers. Given $X, X' \in \partial T_0$, let

$$d_T(Q(X), Q(X')) = \begin{cases} 0, \text{ if } X = X' \\ d_{\overline{T}}(Q(X), Q(X')), \text{ if } X, X' \text{ are } T\text{-bounded} \\ +\infty, \text{ otherwise.} \end{cases}$$

This gives a map $\kappa : (\partial T_0)^2 \times \mathfrak{T}' \to [0, +\infty]$.

Proposition 161 ([34] Proposition 3.8)**.** The map κ is lower-semicontinuous.

Lemma 162. If a very small $(\mathcal{A} * B)$-tree has dense orbits, then all arc stabilizers are trivial.

A proof of this lemma is given by the relative version of Lemma 4.2 in [34] and Proposition 1.10 in [25].

4.3.2 Proof of Theorem 152

Let $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$ be fully irreducible, and with exponential growth. Let $f : \Gamma \to \Gamma$ be a $\mathcal{A}$-Nielsen minimized train track representative for Φ (passing to a power of Φ if necessary).

Proposition 163 ([34] Proposition 5.1)**.** Let T be a minimal $(\mathcal{A} * B)$-tree with dense orbits and trivial arc stabilizers. There exists a leaf $\{X, X'\}$ of Λ^+ or of Λ^- such that $Q(X) \neq Q(X')$.

This proposition follows from the next two lemmas.

Lemma 164 ([34] Lemma 5.2)**.** Suppose $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^+. Let $Y, Y' \in \partial T_0$ belong to the support of Λ^+ (i.e., the complement of some compact subsegment consists of two infinite rays ρ and ρ' contained in Λ^+). Then

$$d_{\overline{T}}(Q(\Phi^i(Y)), Q(\Phi^i(Y'))) \xrightarrow[i \to +\infty]{} 0.$$

Lemma 165 ([34] Lemma 5.3)**.** Let $\widetilde{f} : T \to T$ be the lift of f to the $(\mathcal{A} * B)$-tree T with special points. Suppose $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^-. There exist maps $i_m : T \to \overline{T}$ for all positive integers m such that $i_m \circ \widetilde{f}^m$ is $(\mathcal{A} * B)$-equivariant and

$$\mathrm{BBT}(i_m) \xrightarrow[m \to +\infty]{} 0.$$

Proposition 166. Let T be a minimal $(\mathcal{A} * B)$-tree with a modified very small action. Suppose there exists a simplicial $(\mathcal{A} * B)$-tree T_0, an equivariant map $f : T_0 \to T$ and a geodesic line $\gamma_0 \subset T_0$ representing a leaf of Λ^+ such that $f(\gamma_0)$ has diameter greater than $2\mathrm{BBT}(f)$. Then $f(\gamma_0)$ has infinite diameter and there exists a neighborhood V of $[T] \in \overline{\mathrm{CV}'_n}(\mathcal{A})$ such that $\Phi^m_{|V}$ converges to $[T^+]$ uniformly as $m \to +\infty$.

Proposition 166 is a consequence of the relative version of Lemma 7.1 and Lemma 7.2 in [34]. Notice that the proof of those lemmas are the same also in the relative version and that the proof of Lemma 7.1 is basically the same as that of Lemma 130. We are now ready to prove Theorem 152. We will give a sketch of the proof, for more details see [34].

Proof. Our goal is to show that that every minimal modified very small $(\mathcal{A}*B)$-tree satisfies the hypothesis of Proposition 166 when γ_0 is a leaf of Λ^+. We have to consider three different cases.

1. Suppose T has dense orbits. By Lemma 162, T has trivial arc stabilizers. By Proposition 163, there exists a leaf $\{X, X'\}$ of Λ^+ or of Λ^- such that $Q(X) \neq Q(X')$. By Corollary 154, let $f : T_0 \to T$ with $2\mathrm{BBT}(f) < d(Q(X), Q(X'))$ and let γ_0 be the geodesic joining X and X' in T.

2. Suppose T does not have dense orbits and it is not simplicial. In this case T can be collapsed to a $(\mathcal{A}*B)$-tree T_v with dense orbits. Choose X and X' as in the previous case.

3. Suppose that T is simplicial. In this case it suffices to show that X is T-unbounded for every $\{X, X'\} \in \Lambda^+$.

Given T as in Proposition 166, we have $\lim_{m\to\infty} \Phi^m([T]) = [T^+]$ or $\lim_{m\to\infty} \Phi^m([T]) = [T^-]$. Moreover, since Φ^m converges to $[T^+]$ uniformly on a neighborhood of $[T^+]$ as $m \to \infty$, $[T^+]$ is a stable tree. Similarly, $[T^-]$ is an unstable tree. Now, suppose that $[T] \neq [T^-]$. The set of limit points of the sequence $\Phi^m([T])$ as $m \to +\infty$ is closed and Φ-invariant. Hence, it contains $[T^+]$ or $[T^-]$. Since $[T^-]$ is unstable, it contains $[T^+]$. Therefore, $\lim_{m\to\infty} \Phi^m([T]) = [T^+]$. Similarly, if $[T] \neq [T^+]$, then $\lim_{m\to\infty} \Phi^{-m}([T]) = [T^-]$. $\square$

CHAPTER 5

STRONGLY CONTRACTING GEODESICS IN $\mathrm{CV}'_N(\mathcal{A})$

In this chapter we define the axes in $\mathrm{CV}'_n(\mathcal{A})$ of fully irreducible relative outer automorphisms and the projection of the modified relative outer space to such an axis. The main goal is to prove that these axes are strongly contracting geodesics in $\mathrm{CV}'_n(\mathcal{A})$. The Contracting Geodesics Theorem was proved by Yael Algom-Kfir in [1] for CV_n. Following her approach we prove the relative version of her theorem. We modify the definition of the projection considering primitive basis elements instead of just basis elements. This change affects drastically the proof of some important results (e.g., see Lemma 198 and Proposition 200).

5.1 Axes in $\mathrm{CV}'_n(\mathcal{A})$

As in the case of outer space (see [29]), the definition of an axis of a fully irreducible relative outer automorphism in $\mathrm{CV}'_n(\mathcal{A})$ rely on the concept of fold path.

5.1.1 Fold Paths

A *fold path* in the modified relative outer space is a continuous, proper injection $\gamma : I \to \mathrm{CV}'_n(\mathcal{A})$, where $I \subset \mathbb{R}$ is connected, such that there exists $\gamma(t) = X_t = (\Gamma_t, \phi_t) \in \mathrm{CV}'_n(\mathcal{A})$ for every $t \in I$, and maps $g_{st} : X_t \to X_s$ for $t \leq s$ in I satisfying the following hypothesis:

1. (train track property) g_{st} restricted to each edge in Γ_t is an isometry;

2. (semiflow property) $g_{rt} = g_{rs} \circ g_{st}$ for $r \leq s \leq t$ in I and $g_{tt} : X_t \to X_t$ is the identity for every $t \in I$.

A fold path whose domain is noncompact on both ends is called a *fold line*.

Consider a modified marked metric $(\mathcal{A}, n)$-graph $X = (\Gamma, \psi)$ and two distinct oriented edges e_1, e_2 of $\widehat{\Gamma}$ with the same initial vertex v and positive length. Parameterize e_1, e_2 by arc length from their initial vertex. Choose R such that $e_1([0, R]) \cap e_2([0, R]) = v$. For $r \in [0, R]$ let Γ_r be the graph obtained from Γ by identifying $e_1(t)$ and $e_2(t)$ for each

$t \in [0, 1-e^{-r}]$. Since $e_1(r) \neq e_2(r)$, the quotient map $\Gamma \to \Gamma_r$ is a homotopy equivalence. Pushing forward the metric on Γ by the quotient map and postcomposing the marking of Γ by the quotient map, we get a modified marked metric $(\mathcal{A}, n)$-graph X_r. Notice that we do not require that the relative volume of Γ_r is 1, so Γ_r is in the unprojectivized modified relative outer space. Obviously, the quotient map preserves the marking and is an isometry on the edges of Γ. We say that X_r is obtained from X by a length r fold of e_1 and e_2 and the quotient map is called *fold map*. Rescaling by the volume of Γ_r, we have $X_r \in \mathrm{CV}'_n(\mathcal{A})$.

Consider $\gamma : [0, R] \to \mathrm{CV}'_n(\mathcal{A})$. For $0 \leq r \leq s \leq t \leq R$, the induced map $f_{st} : X_t \to X_s$ satisfies the train track property and the semiflow property. Therefore, γ is a fold path. Combining [24] with this definition of fold map, given $X, Y \in \mathrm{CV}'_n(\mathcal{A})$ it is possible to define a piecewise linear path $\gamma : [0, T] \to \mathrm{CV}'_n(\mathcal{A})$ such that $\gamma(0) = X$ and $\gamma(T) = Y$.

5.1.2 Definition of the Axes

Notice that if $X = (\Gamma, \psi) \in \mathrm{CV}'_n(\mathcal{A})$, $f : \Gamma \longrightarrow \Gamma$ represents $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$ and the loop α_X represents α in X, then $[f(\alpha_X)]$ is an immersed loop in $X \cdot \Phi$ representing α in $X \cdot \Phi$. Let Φ be a fully irreducible relative outer automorphism. Observe that if $f : X \longrightarrow X$ is a train track representative for Φ, then so is $f : X \cdot \Phi \longrightarrow X \cdot \Phi$. Hence, we have a sequence $\{X \cdot \Phi^m\}$ such that $f : X \cdot \Phi^m \longrightarrow X \cdot \Phi^m$ is a train track representative for Φ.

First we construct a path folding X onto itself until we reach $X \cdot \Phi$. In this way we have a fold path $[X, X \cdot \Phi] := \{X_t\}_{0 \leq t \leq \log \lambda}$, where $\log \lambda = d(X, X \cdot \Phi)$. Then we translate this path using Φ to construct a line

$$\mathcal{L}_f = \bigcup_{m=-\infty}^{\infty} [X, X \cdot \Phi] \cdot \Phi^m.$$

This line is automatically invariant under Φ. We will prove that it is a directed geodesic. This construction is described by Handel and Mosher in [29] and by Yael Algom-Kfir in [1] for outer space.

First we start giving the definition of the fold path associated to f and starting at X. Given a train track map $f_0 : X_0 \longrightarrow X_0$ representing Φ, notice that $g_0 : X_0 \longrightarrow X_0 \cdot \Phi$, where $g_0 = f_0$, is a difference in marking. We define a fold line $[X_0, X_0 \cdot \Phi]$ so that for each $0 \leq t \leq \log \lambda$ we will have a map $g_t : X_t \longrightarrow X_0 \cdot \Phi$ which is a difference in marking. We define the path X_t and the maps g_t inductively.

Let e_1, e_2 be two distinct oriented edges in $\widehat{\Gamma_0}$ with the same initial vertex v and positive lengths which define an illegal turn with respect to g_0. Let $R = \max\{r \mid \forall s \leq r : \widehat{g_0}(e_1(s)) = \widehat{g_0}(e_2(s))\}$. Folding e_1 and e_2 as described in the previous section, we get a fold path $\gamma : [0, t_1] \longrightarrow \mathrm{CV}'_n(\mathcal{A})$. Notice that we can uniformly renormalize each graph Γ_t so that its relative volume is 1. In that case, the length of all edges other than the images of e_1, e_2 are scaled by e^t, and the lengths of e_1 and e_2 in Γ_t are $e^t(|e_1|_0 - (1 - e^{-t}))$ and $e^t(|e_2|_0 - (1 - e^{-t}))$ respectively. Moreover, $t_1 = \log\left(\frac{1}{1-R}\right)$. Let $h_t : X_0 \longrightarrow X_t$ be the fold map. Define $g_t : X_t \longrightarrow X_0 \cdot \Phi$ by $g_t(p) = f_0(\tilde{p})$, where $\tilde{p} \in h_t^{-1}(\{p\})$. Note that because we are folding, the choice of $\tilde{p}$ does not matter. Define $f_t : X_t \longrightarrow X_t$ by $f_t(p) = h_t(f_0(\tilde{p}))$. It is easy to verify that f_t is a train track representative for Φ on X_t.

Definition 167. The combinatorial length of a map $h : \Gamma_1 \to \Gamma_2$ that is an isometry on the edges is defined as follows. Subdivide $\widehat{\Gamma_2}$ at the image of the vertices of $\widehat{\Gamma_1}$, which add at most $2n+k-2-2\sum_{i=1}^k s(i)$ new valence 2 vertices to $\widehat{\Gamma_2}$ and at most $2n+k-2-2\sum_{i=1}^k s(i)$ edges, bringing the number of edges in $\widehat{\Gamma_1}$ up to at most $5n + 2k - 5 - 5\sum_{i=1}^k s(i)$. For each edge $e \subset \widehat{\Gamma_1}$ we may regard $\widehat{h}(e)$ as an edge path in the subdivided graph $\widehat{\Gamma_2}$. The *combinatorial length* of h is

$$\sum_{e \subset \widehat{\Gamma_1}} \mathrm{combl}(\widehat{h}(e)).$$

See Section 3.2 for the definition of $\mathrm{combl}(\widehat{h}(e))$.

We continue constructing the path using g_{t_1} instead of g_0. Since the number of foldings of edges is bounded by the combinatorial length of f_0, we will stop after a finite number of steps, obtaining a fold path γ from X_0 to $X_0 \cdot \Phi$.

Notice that since the edges in the wedge cycles have length 0, the fold path γ is well defined in this space. Basically, we are performing the folding as in the case of outer space; the big difference is that now the folding involving at least one edge in a wedge cycle is obtained for free since the length of an edge in the wedge cycles is 0.

Remark 168. If $\widehat{g_t}(p) = \widehat{g_t}(q)$ for some $p, q \in \widehat{\Gamma}_t$ and $\tilde{p}, \tilde{q} \in \widehat{\Gamma_0}$ such that $\widehat{h_t}(\tilde{p}) = p$ and $\widehat{h_t}(\tilde{q}) = q$, then $\widehat{g_0}(\tilde{p}) = \widehat{g_0}(\tilde{q})$, $\widehat{f_0}(\tilde{p}) = \widehat{f_0}(\tilde{q})$ and $\widehat{f_t}(p) = \widehat{f_t}(q)$.

Proposition 169. The fold path $\gamma : [0, \log(\lambda)] \longrightarrow \mathrm{CV}'_n(\mathcal{A})$ is a geodesic parameterized according to arc length.

Proof. As a consequence of Remark 168, if α is legal in Γ_0 with respect to g_0, then it is legal in Γ_0 with respect to h_s and $h_s(\alpha)$ is legal in Γ_s with respect to g_s. Because f is an irreducible train track map, all edges are stretched by the same factor so the tension graph $\Gamma_{g_0} = \widehat{\Gamma_0}$. Moreover, for every fold, all edges are stretched by the same amount. Therefore, $\Gamma_{h_s} = \widehat{\Gamma_0}$ and $\Gamma_{g_s} = \widehat{\Gamma_s}$, hence h_s and g_s are optimal maps. Let α be a legal loop with respect to g_0 with positive length, then α is legal with respect to h_s and $h_s(\alpha)$ is legal with respect to g_s. Thus,

$$d(X_0, X_0 \cdot \Phi) = \log\left(\frac{l(\alpha, X_0 \cdot \Phi)}{l(\alpha, X_0)}\right),\ d(X_s, X_0 \cdot \Phi) = \log\left(\frac{l(\alpha, X_0 \cdot \Phi)}{l(\alpha, X_s)}\right),$$

$$d(X_0, X_s) = \log\left(\frac{l(\alpha, X_s)}{l(\alpha, X_0)}\right).$$

Hence, $d(X_0, X_0 \cdot \Phi) = d(X_0, X_s) + d(X_s, X_0 \cdot \Phi)$. For each fold, at time t, all edges are stretched by e^t, thus the distance from the beginning of the fold X_s to X_{s+t} is $\log e^t = t$. Therefore, $\gamma : [0, \log(\lambda)] \longrightarrow \mathrm{CV}'_n(\mathcal{A})$ is parameterized according to arc length locally and hence globally. $\square$

Definition 170. Let $\Phi \in \mathrm{Out}(F_n; \mathcal{A})$ be a fully irreducible relative outer automorphism, $f : \Gamma_0 \longrightarrow \Gamma_0$ a train track representative, and λ the Perron-Frobenius eigenvalue of Φ. Let $\gamma : [0, \log(\lambda)] \longrightarrow \mathrm{CV}'_n(\mathcal{A})$ be a fold line which starts at X_0 and ends at $X_0 \cdot \Phi$, which is a geodesic parameterized according to arc length. Now we extend γ translating $\gamma([0, \log\lambda])$ by Φ^k and Φ^{-k}. Let $X_t = X_{t-n(t)\log\lambda} \cdot \Phi^{n(t)}$, where $n(t) = \left\lfloor \frac{t}{\log\lambda} \right\rfloor$. Let $\mathcal{L}_f = \{X_t\}_{t\in\mathbb{R}}$. Therefore, we obtain $\gamma : \mathbb{R} \to \mathrm{CV}'_n(\mathcal{A})$ and $\mathrm{Im}(\gamma)$ is a geodesic line invariant under Φ. We say that $\mathcal{L}_f$ is an *axis of the fully irreducible relative outer automorphism* Φ.

Proposition 171. $\mathcal{L}_f$ is a directed geodesic parameterized according to arc length, that is, for $t < t'$ we have $d(X_t, X_{t'}) = t' - t$. In particular, for $t < t' < t''$, $d(X_t, X_{t''}) = d(X_t, X_{t'}) + d(X_{t'}, X_{t''})$.

Proof. Let λ be the expansion factor for f. Then

$$d(X_0, X_0 \cdot \Phi) = \log(\mathrm{Lip}(g_0)) = \log(\mathrm{Lip}(f)) = \log\lambda.$$

Sincef^2 is an optimal difference in marking,

$$d(X_0, X_0 \cdot \Phi^2) = \log(\mathrm{Lip}(f^2)) = \log\lambda^2 = 2\log\lambda.$$

Hence, $d(X_0, X_0 \cdot \Phi^2) = d(X_0, X_0 \cdot \Phi) + d(X_0 \cdot \Phi, X_0 \cdot \Phi^2)$ and this concludes the proof of the proposition. $\square$

5.2 The Projection to an Axis

Let Φ be a fully irreducible relative outer automorphism with exponential growth, and suppose that $f : \Gamma_1 \longrightarrow \Gamma_1$ is an $\mathcal{A}$-Nielsen minimized train track map for Φ and $g : \Gamma_2 \longrightarrow \Gamma_2$ is an $\mathcal{A}$-Nielsen minimized train track map for Φ^{-1}. Let $\mathcal{L}_f = \{\gamma(t)\}_{t\in\mathbb{R}}$ be an axis for Φ, and $\mathcal{L}_g = \{\rho(t)\}_{t\in\mathbb{R}}$ an axis for Φ^{-1}. Let λ, ν be the expansion factors of Φ, Φ^{-1}. We will show that if α is a primitive basis element, then there is a bounded set on which $l(\alpha, \gamma(t))$ achieves its minimum and the bound is uniform over all conjugacy classes α. This will allow us to coarsely define a "nearest point projection" $\pi_f : \mathrm{CV}'_n(\mathcal{A}) \longrightarrow \mathcal{L}_f$. Recall the definition of legality threshold $\kappa = \frac{4\mathrm{BCC}(f)}{\lambda-1}$ given in Section 3.2.

Definition 172. Given a train track map $f : \Gamma \longrightarrow \Gamma$ and a loop α in Γ with positive length, the *legality of* α with respect to the train track structure of f is

$$\mathrm{LEG}_f(\alpha, \Gamma) = \frac{\text{Total length of all legal pieces in } \widehat{\Gamma} \text{ of length } > \kappa}{l(\widehat{\alpha}, \Gamma)}.$$

If $\mathrm{LEG}_f(\alpha, \Gamma) > \epsilon$, then α is called ϵ-*legal*.

Lemma 173. For every $\epsilon > 0$ there is a constant $C = C(\epsilon)$ so that if δ is ϵ-legal, then $l(f^n(\delta), \Gamma) > C\lambda^n l(\delta, \Gamma)$.

Proof. The loop $\delta = a_1 b_1 a_2 \cdots a_s b_s$, where a_i and b_i are paths such that $a_i \in \bigcup \mathbb{B}_j$ and $b_i \in \widehat{\Gamma}$ (a_1 and b_s could be empty). Consider the loop $\widehat{\delta}$ obtained by δ collapsing the wedge cycles to special points. Let $\beta_1, \ldots, \beta_l$ be legal subsegments of $\widehat{\delta}$ of length $> \kappa$. Then $[\widehat{f}(\widehat{\delta})]$ contains $\theta_1, \ldots, \theta_l$ the middle subsegments of $\widehat{f}(\beta_1), \ldots, \widehat{f}(\beta_l)$ after truncating $\mathrm{BCC}(f)$ from either side. Since $l(\theta_i, \Gamma) > \frac{\lambda+1}{2} l(\beta_i, \Gamma)$ for $1 \le i \le l$,

$$\begin{aligned} l(f^n(\delta), \Gamma) \quad > \quad \textstyle\sum_{i=1}^{l} l(\widehat{f}^{n-1}(\theta_i), \Gamma) \quad &> \quad \tfrac{\lambda+1}{2} \textstyle\sum_{i=1}^{l} l(\widehat{f}^{n-1}(\beta_i), \Gamma)) \\ &= \quad \tfrac{\lambda+1}{2} \lambda^{n-1} \textstyle\sum_{i=1}^{l} l(\beta_i, \Gamma) \\ &> \quad \tfrac{\lambda+1}{2\lambda} \lambda^n \epsilon \;\; l(\delta, \Gamma). \end{aligned}$$

Therefore $C(\epsilon) = \frac{\lambda+1}{2\lambda}\epsilon$. ∎

In order to talk about the legality of α we need $l(\widehat{\alpha}, \Gamma) > 0$. Hence, when we consider basis elements in the future we mean basis elements with positive length, that is, not contained in the wedge cycles.

Proposition 174. Let Φ be nongeometric. There is a bound K which depends only on Φ, such that if α is a basis element, then α cannot cross more than K consecutive periodic Nielsen paths in Γ not contained in a wedge cycle.

Proof. We can assume that there is a periodic Nielsen path and its period is one. Since f is $\mathcal{A}$-Nielsen minimized, every indivisible periodic Nielsen path that is not contained in a wedge cycle has period one. By Proposition 93, there exists at most one indivisible Nielsen path $\beta \subset \Gamma$ that is not contained in a wedge cycle.

If β intersects at least one edge of $\widehat{\Gamma}$ exactly once, then the Nielsen path does not close up and hence Φ is nongeometric. Since we have only one Nielsen path and the starting point and the endpoint do not coincide, it is impossible to concatenate Nielsen paths. $\square$

Example 175. Proposition 174 is not true if we suppose that Φ is geometric and α is a non-primitive basis element. Indeed, consider $\mathrm{Out}(F_3; A)$, where $A =< a >$ and $B =< b, c >$. Let Φ be represented by

$$\begin{cases} a & \mapsto & a \\ b & \mapsto & bac \\ c & \mapsto & cbac \end{cases}$$

Note that Φ is geometric because

$$bacb^{-1}c^{-1} \mapsto bacacbacc^{-1}a^{-1}b^{-1}c^{-1}a^{-1}b^{-1}c^{-1} = bacb^{-1}c^{-1}.$$

Let $\beta = bacb^{-1}c^{-1}$. The element β is a basis element. Indeed, $\{ba, c, bacb^{-1}c^{-1}\}$ is a basis for $A * B$. Moreover, $b\beta^K$ is a basis element for any K and $\{b\beta^K, \beta, c\}$ is a basis. Note that the Whitehead graph $\mathrm{Wh}_{\mathcal{B},A}([\beta])$ is connected without a cut vertex.

Proposition 176. There is a bound K which depends only on Φ, such that if α is a primitive basis element, then α cannot cross more than K consecutive periodic Nielsen paths in Γ not contained in a wedge cycle.

Proof. By Proposition 174, we know that this is true if Φ is nongeometric. Hence, let suppose that Φ is geometric. Without loss of generality, we can assume that the period of a Nielsen path is one and the train track map $f : R \to R$, where R is a relative rose.

By Proposition 93, there exists at most one indivisible Nielsen path $\beta \subset \Gamma$ that is not contained in a wedge cycle. Moreover, because Φ is geometric, β intersects each edge of $\widehat{R}$ exactly twice. If α crosses the Nielsen path β more than once, then it crosses all the petals of the rose $\widehat{R}$ twice. Hence, $\mathrm{Wh}_{\mathcal{B},\mathcal{A}}([\alpha])$ is connected and without a cut vertex. By Theorem 102, α is not a primitive element. In conclusion, if α is a primitive basis element, it cannot cross the Nielsen path more than once. $\square$

Lemma 177. For any $\mathcal{A}$-Nielsen minimized irreducible relative outer automorphism $\Phi \in \mathrm{Out}(F_n; A)$ there is a constant $\epsilon_0 > 0$ and an integer N such that for any primitive basis element α,

$$\mathrm{LEG}_f(\Phi^N(\alpha), \Gamma_1) > \epsilon_0 \quad \text{or} \quad \mathrm{LEG}_g(\Phi^{-N}(\alpha), \Gamma_2) > \epsilon_0.$$

Proof. Suppose that Φ is nongeometric. By Lemma 133, one of $\Phi^N(\alpha)_{\Gamma_1}$ and $\Phi^{-N}(\alpha)_{\Gamma_2}$ has a long leaf segment. We must show that one of them has a definite fraction of long leaf segments. By contradiction, suppose that $\{\widehat{\alpha_i}\}$ is a sequence where these fractions converge to 0. Denote by $\tau : R'_{n,k}(\mathcal{A}) \to \Gamma_1$ and $\tau' : R'_{n,k}(\mathcal{A}) \to \Gamma_2$ the markings, and view the α_i's as immersed loop in $R'_{n,k}(\mathcal{A})$. We can find the segments in α_i of arbitrarily long length that after applying $\tau\Phi^N$ and $\tau'\Phi^{-N}$ and thightening do not contain legal segments of length $\geq C$ except within a bounded distance from the endpoints. Choosing a subsequence and limiting produces an immersed line in $\widehat{R_{n,k}}(\mathcal{A})$ whose $\widehat{\tau}$-image in $\widehat{\Gamma_1}$ violates Lemma 133.

We use the assumption that Φ is nongeometric only to bound the number of consecutive Nielsen paths appearing in $\widehat{\alpha}$. By Proposition 176 we have such a bound for primitive basis elements. This concludes the proof of the lemma. $\square$

Fix a conjugacy class of a primitive basis element α. Notice that if $\mathrm{LEG}_f(\Phi^N(\alpha), \Gamma_1) \geq \epsilon$, then $\mathrm{LEG}_f(\Phi^m(\alpha), \Gamma_1) \geq \epsilon$ for all $m > N$. Define

$$\begin{aligned} k_0 &= \max\{m | \mathrm{LEG}_f(\Phi^m(\alpha), \Gamma_1) < \epsilon_0\} \\ k_0' &= \min\{m | \mathrm{LEG}_g(\Phi^m(\alpha), \Gamma_2) < \epsilon_0\}. \end{aligned}$$

Recall that for all primitive basis elements α, the weak limit $\lim_{n \longrightarrow \infty} \Phi^n(\alpha)$ is the stable lamination and the weak limit $\lim_{n \longrightarrow -\infty} \Phi^n(\alpha)$ is the unstable lamination (see Chapter 4). Therefore, there is some m such that $\Phi^m(\alpha)$ is ϵ_0-legal and $\Phi^{-m}(\alpha)$ is not ϵ_0-legal in Γ_1. The same argument applied to g shows the existence of k_0'. At k_0, $\mathrm{LEG}_f(\Phi^{k_0}(\alpha), \Gamma_1) \not> \epsilon_0$, thus by Lemma 177, $\mathrm{LEG}_g(\Phi^{k_0 - 2N}(\alpha), \Gamma_2) > \epsilon_0$. This implies that $k_0' > k_0 - 2N$. By the symmetry of f and g we get the following result.

Lemma 178. There is an N so that for all primitive basis element α, $|k_0(\alpha) - k_0'(\alpha)| < N$.

Definition 179. Let $t_0(\alpha) = k_0 \log \lambda$ and $t_0'(\alpha) = k_0' \log \lambda$.

Lemma 180. There exists a C such that for every primitive basis element α if $n(t) = \lfloor \frac{|t|}{\log \lambda} \rfloor$, then for $t > 0$

$$\frac{1}{C} \cdot \lambda^{n(t)} \cdot l(\alpha, \gamma(t_0)) < l(\alpha, \gamma(t_0 + t)) \leq C \cdot \lambda^{n(t)} \cdot l(\alpha, \gamma(t_0)) \tag{5.1}$$

and for $t < 0$

$$\frac{1}{C} \cdot \mu^{n(t)} \cdot l(\alpha, \gamma(t_0)) < l(\alpha, \gamma(t_0 + t)) \leq C \cdot \mu^{n(t)} \cdot l(\alpha, \gamma(t_0)). \tag{5.2}$$

Proof. The right-hand inequality of (5.1) is obvious. Because α is ϵ_0-legal at $t_0 + \log\lambda$, by Lemma 173 (for $\epsilon = \epsilon_0$)

$$\begin{aligned} & l(\alpha, \gamma(t_0+t)) > \lambda^{-1} l(\alpha, \gamma(t_0 + n(t)\log\lambda)) = \\ = \ & \lambda^{-1} l(\Phi^{n(t)}(\alpha), \gamma(t_0)) > \lambda^{-1} C \lambda^{n(t)} l(\alpha, \gamma(t_0)). \end{aligned}$$

To prove the inequalities in (5.2), first note that

$$l(\alpha, \gamma(t_0+t)) \leq \mu \cdot l(\Phi^{-n(t)}(\alpha), \gamma(t_0)).$$

Let $D = \max\{d(\gamma(t_0), \rho(t_0')), d(\rho(t_0'), \gamma(t_0))\}$. Note that D is bounded independently of α because $|k_0 - k_0'| < N$ and (by periodicity) $d(\gamma(t), \rho(t))$ is bounded independently of t. Let $K = e^D$. Thus,

$$\mu^{n(t)} l(\alpha, \gamma(t_0)) \geq \frac{1}{K} \mu^{n(t)} l(\alpha, \rho(t_0')) \geq \frac{1}{K} l(\Phi^{-n(t)}(\alpha), \rho(t_0')) \geq$$

$$\geq \frac{1}{K^2} l(\Phi^{-n(t)}(\alpha), \gamma(t_0)) \geq \frac{1}{\mu K^2} l(\alpha, \gamma(t_0+t)).$$

We get the right-hand inequality in (5.2). The left-hand inequality is proven similarly. $\square$

Definition 181. For a conjugacy class α of a primitive basis element, let $L = \min\{l(\alpha, \gamma(t)) \mid t \in \mathbb{R}\}$ and denote by T_α the set of t_α such that $l(\alpha, \gamma(t_\alpha)) = L$. The *min set of* α is $\pi_f(\alpha) = \{\gamma(t_\alpha) \mid t_\alpha \in T_\alpha\}$.

It follows from Lemma 180 the following results.

Corollary 182. There exists a constant $s > 0$ so that for any primitive basis element α and for all $t_\alpha \in T_\alpha$, $|t_\alpha - t_0| < s$.

Corollary 183. There is a constant $s > 0$ such that for every primitive basis element α, $\operatorname{diam}\{T_\alpha\} < s$. Hence, $\operatorname{diam}\{\pi_f(\alpha)\}$ is bounded independently of α.

From now on t_α denotes any choice of element in T_α, for example the smallest one. The following corollary states that the min sets of α with respect to $\mathcal{L}_f$ and $\mathcal{L}_g$ are uniformly close and it follows from Corollary 182 and Lemma 178.

Corollary 184. There is an $s > 0$ such that for every primitive basis element α, for any $\Gamma_1' \in \pi_f(\alpha)$ and any $\Gamma_2' \in \pi_g(\alpha)$, $d(\Gamma_1', \Gamma_2') < s$.

Corollary 185. There is an $s > 0$ such that for every primitive basis element α, if $t > t_\alpha + s$ then $\mathrm{LEG}(\alpha, \gamma(t)) > \epsilon_0$, where the legality is computed with respect to the train track structure induced by $g_t : \Gamma_t \longrightarrow \Gamma_t$.

The following observation states that if α is almost legal in $\gamma(t)$, then it almost realizes the distance $d(\gamma(t), \gamma(t+t'))$.

Proposition 186. There is a C so that if α is ϵ_0-legal in $\gamma(t)$ with respect to g_t then for all $t' > 0$,

$$\log \mathrm{St}_\alpha(\gamma(t), \gamma(t+t')) - C \geq d(\gamma(t), \gamma(t+t')) = t'.$$

Proof. Since α is ϵ_0-legal, $t > t_0$. Let C be the constant from Lemma 180. Then

$$\begin{aligned} l(\alpha, \gamma(t+t')) &\geq \tfrac{1}{C}\lambda^{n(t+t'-t_0)} l(\alpha, \gamma(t_0)) \\ l(\alpha, \gamma(t)) &\leq C\lambda^{n(t-t_0)} l(\alpha, \gamma(t_0)). \end{aligned}$$

Hence,

$$\mathrm{St}_\alpha(\gamma(t), \gamma(t+t')) = \frac{l(\widehat{\alpha}, \gamma(t+t'))}{l(\widehat{\alpha}, \gamma(t))} \geq \frac{1}{C^2}\lambda^{n(t+t'-t_0)-n(t-t_0)}.$$

□

Now we can define a coarse projection $\pi_f : \mathrm{CV}'_n(\mathcal{A}) \longrightarrow \mathcal{L}_f$.

Definition 187. Let $X \in \mathrm{CV}'_n(\mathcal{A})$ and $T_X = \{t \mid d(X, \gamma(t)) = d(X, \mathcal{L}_f)\}$. Define the projection of X to $\mathcal{L}_f$ by $\pi_f(X) = \{\gamma(t) \mid t \in T_X\}$.

Proposition 188. There is a constant $s > 0$ such that for every point $X \in \mathrm{CV}'_n(\mathcal{A})$,

$$\mathrm{diam}(\pi(X)) < s.$$

Proof. Let α, β be two candidates in X and consider $u(t) = \mathrm{St}(\alpha_t)$ and $v(t) = \mathrm{St}(\beta_t)$. Note that they satisfy Lemma 180. The function $h(t) = \max\{u(t), v(t)\}$ has a coarse minimum. A proof of this fact can be found in [1]. Since there is only a finite number of classes of candidates, the diameter of $\pi(X)$ is uniformly bounded. □

5.3 The Morse Lemma

Let X be a metric space with an asymmetric metric. Suppose that for any two points $a, b \in X$ there is a geodesic $[a, b]$ connecting a to b. Since the space has an asymmetric metric, when we compute the distance between two points it is very important to specify the order of the points. Let A be a set. The δ-neighborhood of A is

$$N_\delta(A) = \{x \in X \mid d(a, x) < \delta \text{ for some } a \in A\}.$$

Let $r > 0$. The ball of radius r centered at x is

$$B_x(r) = \{y \in X \mid d(x, y) < r\}.$$

Definition 189 (Strongly contracting geodesics in an asymmetric space)**.** Let L be a directed geodesic in X, and let $\pi_L : X \longrightarrow L$ be the closest point projection. L is D-*strongly contracting* if for any ball $B_x(r) \subseteq X$ disjoint from L, $\mathrm{diam}(\pi_L(B_x(r))) < D$.

We will prove in Theorem 211 that if Φ is a fully irreducible relative outer automorphism and $\mathcal{L}_\Phi$ an axis for Φ, then there is a $D = D(\Phi) > 0$ so that $\mathcal{L}_\Phi$ is D-strongly contracting.

Definition 190. The map $\alpha : [0, l] \longrightarrow X$ is a directed (k, c)-*quasi-geodesic* if for all $0 \leq t_1 < t_2 \leq l$ we have

$$\frac{1}{k}(t_2 - t_1) - c \leq d(\alpha(t_2), \alpha(t_1)) \leq k(t_2 - t_1) + c.$$

Definition 191. A quasi-geodesic $\alpha : [0, l] \longrightarrow X$ is (m, p)-*tame* if for all $0 \leq t_1 < t_2 \leq l$ we have

$$\mathrm{len}(\alpha|_{[t_1,t_2]}) \leq m(t_2 - t_1) + p.$$

Lemma 192. For every (k, c)-quasi-geodesic $\alpha : [0, l] \longrightarrow X$ there is an (m, p)-tame (k', c')-quasi-geodesic $\beta : [0, l] \longrightarrow X$ with

1. $\beta(0) = \alpha(0), \beta(l) = \alpha(l)$;
2. $k' = k$, $c' = 2(k + c)$;
3. $m = k(k + c)$, and $p = (k + c)(2k^2 + 2kc + 3)$;
4. $N_{k+c}(\mathrm{Im}\alpha) \supseteq \mathrm{Im}\beta$ and $N_{k+c}(\mathrm{Im}\beta) \supseteq \mathrm{Im}\alpha$.

The proof of Lemma 192 for a metric space with a symmetric metric can be found in [16]. The proof for a nonsymmetric space is the same so we omit it.

Definition 193. A point $x \in X$ is A-*high* (or just *high*) if there exists a constant A such that $d(x, y) \leq A \cdot d(y, x)$ for all $y \in X$. A set $S \in X$ is high if there are constants A so that for all $x \in S$ and $y \in X$, $d(x, y) \leq A \cdot d(y, x)$.

We recall the definition of Hausdorff distance.

Definition 194. Let $S, T \subset X$ be closed. Define the Hausdorff distance

$$d_{\mathrm{Haus}}(S, T) = \inf\{\varepsilon \mid S \subseteq N_\varepsilon(T) \text{ and } T \subseteq N_\varepsilon(S)\}.$$

Theorem 195 (Morse Lemma)**.** If L is a directed, A-high, D-strongly contracting geodesic in X and and α is an (a, b)-quasi-geodesic with endpoints on L then there exists a constant $C = C(A, D, a, b)$, such that $d_{\mathrm{Haus}}(\mathrm{Im}L, \mathrm{Im}\alpha) < C$.

Remark 196. The Morse Lemma is still true if we suppose that α satisfies: $\text{len}(\alpha|_{[t_1,t_2]}) < a \cdot d(\alpha(t_1), \alpha(t_2)) + b$.

See [1] for a proof of the Morse Lemma.

5.4 Bounds on the Projection

In the next sections of this chapter we will use the same notation for laminations introduced in Chapter 4.

Definition 197. Let η be a leaf of Λ^+ or Λ^- in X. Let γ be an edge path contained in η. We say that γ is an *r-piece* of η if $l(\widehat{\gamma}, X) \geq r$.

If $H < F_n$, we denote by $[H]$ its conjugacy class. The next proposition states that primitive basis elements cannot contain long pieces of both Λ^+ and Λ^-.

Lemma 198. Let Λ_1 and Λ_2 be two minimal laminations such that $\Lambda_1 \cap \Lambda_2 = \emptyset$ (i.e., they do not have any line in common) and neither of them is carried by a free factor. Then there is a constant $\jmath > 0$ such that the following statements are true.

1. If β is a primitive basis element of $\mathcal{A} * B$ represented by an immersed loop with positive length, which we shall also denote by β in Γ, then there do not exist leaves $\ell_1 \in \Lambda_1(\Gamma)$ and $\ell_2 \in \Lambda_2(\Gamma)$ such that β contains a $\jmath$-piece of ℓ_1 or the inverse of a $\jmath$-piece of ℓ_1 and a $\jmath$-piece of ℓ_2.

2. If α, β are tight loops in Γ with positive length corresponding to primitive basis elements and $\alpha, \beta, y_1^1, \ldots, y_{s(k)}^k$ are compatible, then there do not exist leaves $\ell_1 \in \Lambda_1(\Gamma)$ and $\ell_2 \in \Lambda_2(\Gamma)$ such that α contains a $\jmath$-piece of ℓ_1 or the inverse of a $\jmath$-piece of ℓ_1 (a $\jmath$-piece of ℓ_2 or the inverse of a $\jmath$-piece of ℓ_2) and β contains a $\jmath$-piece of ℓ_2 (a $\jmath$-piece of ℓ_1 or the inverse of a $\jmath$-piece of ℓ_1).

Proof. 1. By contradiction, suppose that $\{\beta_m\}$ is a sequence of primitive basis elements containing segments σ_j of length $> m$ from each $\Lambda_j(R_m)$, $j = 1, 2$, where R_m is a (relative) rose such that β_m corresponds to one petal P_m in R_m and the loops corresponding to the wedge cycles have length 0. Let $\varphi^{-1} : R_m \to R'_{n,k}(\mathcal{A})$ be a homotopy inverse of the marking of R_m. Let φ^{-1} be so that it takes vertices to vertices and it is an immersion on every edge. We put a metric on each edge of R_m pulling back the metric on $R'_{n,k}(\mathcal{A})$ under the map φ^{-1} so that for each edge $e \subset R_m$ we have

$$l(e, R_m) = l(\varphi^{-1}(e), R'_{n,k}(\mathcal{A})).$$

Notice that because $\Lambda_1 \cap \Lambda_2 = \emptyset$, there exists a constant K such that every segment that appears in the realization in R_m of any line of $\Lambda_1 \cup \Lambda_2$ has length less than K. Otherwise, taking the limit of that segment we get a leaf which is in both laminations. From this observation it follows that for each path γ in R_m with $l(\gamma, R_m) > K$ there exists $i \in \{1, 2, \}$ such that γ is not a subpath of the realization in R_m of any line in Λ_i. Now we start folding the edges in P_m till β_m is represented by a loop (see Figure 5.1).

Let Γ be the graph obtained by folding, let S_m be the loop corresponding to β_m and $\psi^{-1} : \Gamma \to R'_{n,k}(\mathcal{A})$. Suppose $m >> 2K$ so that $l(S_m, \Gamma) > 2K$. Now, factor $\psi^{-1} : \Gamma \to R'_{n,k}(\mathcal{A})$ as a composition of folds

$$\Gamma = \Gamma_0 \xrightarrow{f_1} \cdots \xrightarrow{f_N} \Gamma_N = R'_{n,k}(\mathcal{A}).$$

Let $G_j = f_N \circ f_{N-1} \circ \cdots \circ f_{j+1} : \Gamma_j \to R'_{n,k}(\mathcal{A})$ and $F_j = f_j \circ f_{j-1} \circ \cdots \circ f_1 : R \to \Gamma_j$. Hence, $\psi^{-1} = G_j \circ F_j$. Let $I \in \{1, \ldots, N\}$ be the initial value such that $l(e, \Gamma_I) \leq 2K$, $\forall e \subset \Gamma_I$. Note that some edge $e_I \subset \Gamma_I$ has length $l(e_I, \Gamma_I) > K$. Otherwise, using that the fold map $f_I : \Gamma_{I-1} \to \Gamma_I$ takes each edge of $\widehat{\Gamma_{I-1}}$ to a path of at most two edges in $\widehat{\Gamma_I}$, each edge of $\widehat{\Gamma_{I-1}}$ would have length $\leq 2K$ contradicting the minimality of I. By the previous remark, there exists $i \in \{1, 2\}$, let say $i = 1$, such that e_I is not a subpath of the realization in Γ_I of any line in Λ_1. Notice that e_I cannot be a separating edge. Otherwise the lines in Λ_1 would be disconnected. Moreover, since $l(e_I, \Gamma_I) > K > 0$, e_I is not contained in a wedge cycle. Therefore e_I is a non-separating edge which is not contained in a wedge cycle. Consider the subgraph $H = \Gamma_I \setminus e_I$. Folding the edges not in S_m first, we can suppose that $e_I \subset F_I(S_m)$ has length $> K$ and F_I is an immersion on H. Let F be the free factor containing

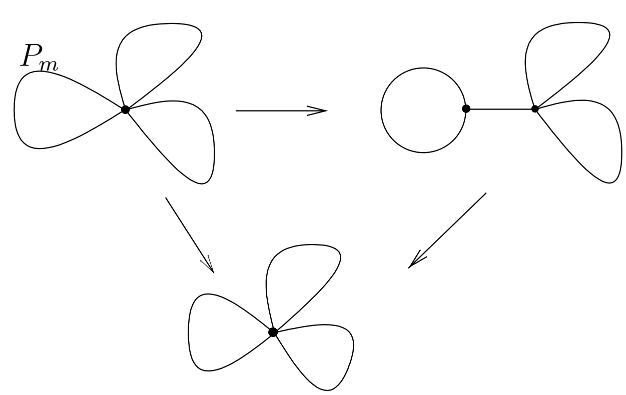

Figure 5.1. Folding the edges in P_m.

$[\mathcal{A}]$ such that $[H] = [F]$. Since the number of proper free factors containing $[\mathcal{A}]$ is finite and the sequence $\{\beta_m\}$ is infinite, there exists an (infinite) subsequence $\{\beta_{r_j}\}$ such that $[H] = [F]$ for the same free factor F. Without loss of generality we can suppose that e_I is not a subpath of the realization in Γ_I of any line in Λ_1. Consider the triples $(\Gamma, G, \varphi^{-1})$ consisting of a modified relative marked graph Γ, a proper connected subgraph $G \subset \Gamma$ with no valence 1 vertices except for the special points corresponding to the wedge cycles, and a map $\psi^{-1} : \Gamma \to R'_{n,k}(\mathcal{A})$ which is a homotopy inverse of the marking of Γ, such that ψ^{-1} takes vertices to vertices, wedge cycles to wedge cycles, it is an immersion on each edge of Γ, and it is an immersion on G. Two triples $(\Gamma_1, G_1, \varphi_1^{-1}), (\Gamma_2, G_2, \varphi_2^{-1})$ are equivalent if there exists a homeomorphism $h : (\Gamma_1, G_1) \to (\Gamma_2, G_2)$ that fixes the wedge cycles and such that $\varphi_2^{-1} \circ h = \varphi_1^{-1}$. For each $l \in \Lambda_1 \cup \Lambda_2$ let $U(l, C) \subset \mathcal{B}(R'_{n,k}(\mathcal{A}))$ be the weak neighborhood of l consisting of lines in $R'_{n,k}(\mathcal{A})$ that have a subpath which is equal to the subpath of edge of length $2C$ centered on the base point. Since Λ_i is not carried by a free factor $(i = 1, 2)$, we may choose $l_i(F) \in \Lambda_i$ which is not carried by F. Since the set of lines carried by F is closed, we may choose a constant $C_i(F) > 0$ such that the neighborhood $U(l_i(F), C_i(F))$ of $l_i(F)$ does not contain any line that is carried by F. The same is true if we replace $C_i(F)$ by $C \geq C_i(F)$. By applying the Bounded Cancelation Lemma to the map φ^{-1}, letting

$$C_i(\Gamma_I, \Gamma_I \setminus e_I, F_I) = C_i(F) + \mathrm{BCC}(\varphi^{-1})$$

and using the fact that any path in S_{r_j} extends to a line in S_{r_j}, we may ensure that for any finite path $\sigma \subset S_{r_j}$ no subpath of $\varphi^{-1}(\sigma)$ is equal to the base point centered subpath of $l_1(F)$ with edge length $2C_1(\Gamma_I, \Gamma_I \setminus e_I, F_I)$. Let $C(\Gamma_I, \Gamma_I \setminus e_I, F_I) = \max_{i=1,2} C_i(\Gamma_I, \Gamma_I \setminus e_I, F_I)$. Define

$$C = 4K + \max\{C(\Gamma_I, \Gamma_I \setminus e_I, F_I)\},$$

where the maximum is taken over the finitely many equivalence classes of triples $(\Gamma_I, \Gamma_I \setminus e_I, F_I)$ for which $l(e, \Gamma_I) \leq 2K$, for each edge $e \subset \widehat{\Gamma_I}$ and $S_{r_i} = S_{r_j}$. Note that C depends only on Λ_1 and Λ_2. Now, let σ be the realization in S_{r_j} of some line in $U(l_1(F), C)$ contained in $\Gamma_I \setminus e_I$. Since $\varphi^{-1}_{|\sigma}$ is an immersion, $F_{I|\sigma}$ and $G_{I|F_I(\sigma)}$ are immersions. If r_j is big enough, there is a finite subpath $\sigma_0 \subset \sigma$ (with endpoints not necessarily at vertices) such that $\varphi^{-1}(\sigma_0)$ is the base point centered subpath of $l_1(F)$ in $R'_{n,k}(\mathcal{A})$ with $l(\varphi^{-1}(\sigma_0), R'_{n,k}(\mathcal{A})) = 2C$. Thus $F_I(\sigma_0)$ does not contain e_I as a subpath. It follows that there is a subpath $\sigma'_0 \subset \sigma_0$ such that $F_I(\sigma'_0)$ is contained in $\Gamma_I \setminus e_I$ and $\varphi^{-1}(\sigma_0') = G_I(F_I(\sigma_0'))$ is the base point centered subpath of $l_1(F)$ in $R'_{n,k}(\mathcal{A})$) obtained from $\varphi^{-1}(\sigma_0)$ by removing initial and/or terminal segments of length less than $2K$. We have

$$\begin{aligned} 2C = l(\varphi^{-1}(\sigma_0), R'_{n,k}(\mathcal{A})) &< l(\varphi^{-1}(\sigma_0'), R'_{n,k}(\mathcal{A})) + 4K \\ &< 2C_1(\Gamma_I, \Gamma_I \setminus e_I, F_I) + 4K \\ &\leq 2C(\Gamma_I, \Gamma_I \setminus e_I, F_I) + 4K \\ &\leq C. \end{aligned}$$

2. The proof of the second claim is similar to (1). Let $\{\alpha_m, \beta_m\}$ be a sequence of primitive basis elements such that

   - $\alpha, \beta, y_1^1, \ldots, y_{s(k)}^k$ are compatible,
   - α_m contains a m-piece of a leaf in $\Lambda_1(R_m^{\alpha,\beta})$ and β_m contains a m-piece of a leaf in $\Lambda_2(R_m^{\alpha,\beta})$, where $R_m^{\alpha,\beta}$ is a rose such that α_m and β_m correspond to petals in the rose $R_m^{\alpha,\beta}$.

   Applying the same argument as in (1) we reach a contradiction.

$\square$

Example 199. Consider $[\varphi] \in \mathrm{Out}(F_3; A)$, where $A =< a >$, $B =< b_1, b_2 >$, and

$$\varphi : \begin{cases} a \mapsto a \\ b_1 \mapsto b_1 b_2 \\ b_2 \mapsto b_2 b_1 b_2 \end{cases}$$

In this case the laminations Λ_1 and Λ_2 are carried by the free factor $F = B =< b_1, b_2 >$.

Proposition 200. There exists a constant $\jmath > 0$ so that for all $X_t \in \mathcal{L}_f$ the following statements are true.

1. If β is a primitive basis element of $\mathcal{A} * B$ represented by an immersed loop with positive length, which we shall also denote by β in Γ_t, then there do not exist leaves $\ell_1 \in \Lambda_f^+(\Gamma_t)$ and $\ell_2 \in \Lambda_f^-(\Gamma_t)$ such that β contains a $\jmath$-piece of ℓ_1 or the inverse of a $\jmath$-piece of ℓ_1 and a $\jmath$-piece of ℓ_2.

2. If α, β are tight loops in Γ_t with positive length corresponding to primitive basis elements and $\alpha, \beta, y_1^1, \ldots, y_{s(k)}^k$ are compatible, then there do not exist leaves $\ell_1 \in \Lambda_f^+(\Gamma_t)$ and $\ell_2 \in \Lambda_f^-(\Gamma_t)$ such that α contains a $\jmath$-piece of ℓ_1 or the inverse of a $\jmath$-piece of ℓ_1 (a $\jmath$-piece of ℓ_2 or the inverse of a $\jmath$-piece of ℓ_2) and β contains a $\jmath$-piece of ℓ_2 (a $\jmath$-piece of ℓ_1 or the inverse of a $\jmath$-piece of ℓ_1).

Proof. Let $\Lambda_1 = \Lambda_f^+(\Gamma_t)$ and $\Lambda_2 = \Lambda_f^-(\Gamma_t)$. Since f is a representative of a fully irreducible relative outer automorphism, Λ_1 and Λ_2 are minimal and $\Lambda_1 \cap \Lambda_2 = \emptyset$. If neither of the laminations is carried by a free factor, then by Lemma 198, there exist $\jmath > 0$ with those properties. If not, it means that there exist $y_{i_1}, \dots, y_{i_r}$ in $\{y_1^1, \dots, y_{s(k)}^k\}$ such that each leaf in $\Lambda_1 \cup \Lambda_2$ is not crossing the cycles $C_{i_1}, \dots C_{i_r}$ corresponding to $y_{i_1}, \dots, y_{i_r}$ (see Example 199). Let $Y =< y_{i_1}, \dots, y_{i_r} >$ and $\mathcal{A}' = \mathcal{A}/Y$. Suppose that such a constant $\jmath$ does not exist. Let consider the first statement and the sequence $\{\beta_m\}$ and the rose R_m as in Lemma 198. Collapsing the loops corresponding to $C_{i_1}, \dots C_{i_r}$, we get a new rose R'_m. Since β is compatible with $y_1^1, \dots, y_{s(k)}^k$, the loop β is not collapsed. We can now consider $\mathrm{Out}(F_{n-r}; \mathcal{A}')$ and $\mathrm{CV}'_{n-r,k-r}(\mathcal{A}')$. Since the laminations Λ_1, Λ_2 restricted to the new graphs are still minimal, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and they are not carried by a free factor in $\mathcal{A}' * B$, following the proof of Lemma 198, we reach a contradiction. The same proof works for the second statement of the proposition considering the rose $R_m^{\alpha,\beta}$ instead of R_m. $\square$

An important application of Proposition 200 is the following lemma.

Lemma 201. There is an $s > 0$ such that if α, β are primitive basis elements which are compatible, then $|t_\alpha - t_\beta| < s$.

Proof. Denote $t_1 = t_\alpha, t_2 = t_\beta$. Suppose $t_2 > t_1$. Let α_t represent α in Γ_t, and β_t represent β in Γ_t. We claim that there is a t_0 such that if $t < t_2 - t_0$ then β_t contains a $\jmath$-piece of $(\ell_2)_{\Gamma_t}$, and if $t > t_1 + t_0$ then α_t contains a $\jmath$-piece of $(\ell_1)_{\Gamma_t}$. By Proposition 185, there is an s_1 such that if $t > t_1 + s_1$, then $\mathrm{LEG}_f(\alpha_t, \gamma(t)) > \epsilon_0$. Let $\alpha'_t \subseteq \alpha_t$ be a legal segment of length $> \kappa$ (the legality threshold, see Section 3.2). There is an N such that $f^N(\alpha'_t)$ is longer than $\frac{2\jmath}{\lambda+1}$, where λ is the expansion factor of f. By the definition of legality threshold, $[f_t^N(\alpha_t)]$ will contain a $\jmath$-piece of the lamination contributed from $f_t^N(\alpha'_t)$. Let $s_2 = s_1 + N\log(\lambda)$. Now, at $t_0 = t_1 + s_2$, α contains a $\jmath$-piece of ℓ_1, contributed by α'_t. Because $\mathcal{L}_f$ and $\mathcal{L}_g$ are close t_α and $t_{\alpha'}$ are close by Corollary 184 a similar statement is true for g. Therefore, if $|t_2 - t_1| > 2t_0$ and $r = t_1 + t_0$, then α_r contains an $\jmath$-piece of $(\ell_1)_{\Gamma_r}$ and β_r contains a $\jmath$ piece of $(\ell_2)_{\Gamma_r}$ which contradicts Proposition 200. $\square$

Corollary 202. There exists a constant $s > 0$ such that if α and β are candidates in X, then $|t_\alpha - t_\beta| < s$. Moreover, $|t_X - t_\alpha| < s$.

Proof. By Proposition 103, there is a primitive basis element γ so that α, γ and γ, β can be completed to a basis of $\mathcal{A} * B$. By Lemma 201 there is an s such that $|t_\alpha - t_\gamma| < s$ and $|t_\gamma - t_\beta| < s$. Thus $|t_\alpha - t_\beta| < 2s$.

Let $\alpha_1, \ldots, \alpha_N$ be the classes of candidates of X. Then for each i, $\min \mathrm{St}_{\alpha_i}(X, \gamma(t)) = \mathrm{St}_{\alpha_i}(X, \gamma(t_{\alpha_i}))$. By the proof of Proposition 103, the minimum of

$$h(t) = \max\{\mathrm{St}_{\alpha_i}(X, \gamma(t)), \mathrm{St}_{\alpha_j}(X, \gamma(t))\}$$

is realized by a point in $[\min\{t_{\alpha_i}, t_{\alpha_j}\}, \max\{t_{\alpha_i}, t_{\alpha_j}\}]$. By induction, the minimum of $d(X, \gamma(t)) = \max\{\mathrm{St}_{\alpha_i}(X, \gamma(t)) \mid 1 \leq i \leq N\}$ is realized at $t = t_X$ in $[\min\{t_{\alpha_i} \mid 1 \leq i \leq N\}, \max\{t_{\alpha_i} \mid 1 \leq i \leq N\}]$. By the previous remark, $|t_X - t_\alpha| < s$. □

Corollary 203. There exists an $s > 0$ such that if the translation length of a primitive $\alpha \in \mathcal{A} * B$ in both X and Y is positive but smaller than 1, then $|\pi(X) - \pi(Y)| < s$.

We say that a basis $< x_1, x_2, \ldots, x_n >$ of $\pi_1(X)$ is *short* if all the loops have length smaller than 1.

Proof. Let $< y_1^1, \ldots, y_{s(k)}^k, \alpha_1, \ldots, \alpha_{n-\sum s(i)} >$ be a short basis for $\pi_1(X)$, and $< y_1^1, \ldots, y_{s(k)}^k$, $\beta_1, \ldots, \beta_{n-\sum s(i)} >$ a short basis for $\pi_1(Y)$. Since the relative volume of X is 1, α is primitive, and the translation length of α is positive, α is carried by a free factor not contained in $\mathcal{A}$, e.g. $< y_{i_1}, \ldots y_{i_l}, \alpha_1, \ldots, \alpha_r >$ with $r < n - \sum s(i)$ and at least one element of $\{\alpha_{r+1}, \ldots, \alpha_{n-\sum s(i)}\}$ is primitive. Similarly, because the relative volume of Y is 1, α is primitive, and the translation length of α is positive, α is carried by a free factor not contained in $\mathcal{A}$, e.g. $< y_{j_1}, \ldots y_{j_p}, \beta_1, \ldots, \beta_r >$ with $r < n - \sum s(i)$ and at least one element of $\{\beta_{r+1}, \ldots, \beta_{n-\sum s(i)}\}$ is primitive. Suppose that α_{r+1} and β_{r+1} are primitive basis elements. By Lemma 201 there exists s such that $|t_\alpha - t_{[\alpha_{r+1}]}| < s$. Similarly, for Y, $|t_\alpha - t_{[\beta_{r+1}]}| < s$. So $t_{[\alpha_{r+1}]}$ and $t_{[\beta_{r+1}]}$ are uniformly close. By Corollary 202, we have that t_X and t_Y are uniformly close. □

Corollary 203 shows that if α is a primitive element, $\pi_f(\{X \in \mathrm{CV}'_n(\mathcal{A}) \mid l(\alpha, X) < 1\})$ is a bounded interval of $\mathcal{L}_f$.

5.5 Axes are Contracting

The main goal of this section is to prove Theorem 211: the axes defined in Section 5.1 are strongly contracting. We will follow the same approach as in [1].

Lemma 204. There exist constants $s, c > 0$ such that for any Y, if $|t - t_Y| > s$, then $d(Y, \gamma(t)) \geq d(Y, \pi(Y)) + d(\pi(Y), \gamma(t)) - c$.

Proof. Denote $X = \gamma(t)$. Let us first prove it for $t > t_Y$. By Lemma 202 there exists a constant s_1 such that for all candidates α of Y, $|t_\alpha - t_Y| < s_1$. By Lemma 185 there exists a constant s_2 such that if $t > t_\alpha + s_2$, then $\mathrm{LEG}_f(\alpha_t, \gamma(t)) > \epsilon_0$. Let $Z = \gamma(t_Y + s_1 + s_2)$. Then for any candidate β of Y, $\mathrm{LEG}_f(\beta, Z) > \epsilon_0$. Now suppose that β_Y in Y is a loop that realizes $d(Y, Z)$, that is, $\mathrm{St}_\beta(Y, Z) = e^{d(Y,Z)}$. Since β is ϵ_0-legal in Z, by Corollary 186 there is a constant C so that $\mathrm{St}_\beta(Z, X) \geq Ce^{d(Z,X)}$. Hence,

$$\mathrm{St}(Y, X) \geq \mathrm{St}_\beta(Y, X) = \mathrm{St}_\beta(Y, Z)\mathrm{St}_\beta(Z, X) \geq Ce^{d(Y,Z)+d(Z,X)}.$$

Thus $d(Y, X) \geq \log(C) + d(Y, Z) + d(Z, X)$. Now, recall that $Z = \gamma(t_Y + s_1 + s_2)$ so $d(\pi(Y), Z) = s_1 + s_2$. We have, $d(Y, \pi(Y)) \leq d(Y, Z)$ because π is a projection. Moreover,

$$d(\pi(Y), X) \leq d(\pi(Y), Z) + d(Z, X) = (s_1 + s_2) + d(Z, X)$$

and hence $d(Z, X) > d(\pi(Y), X) - (s_1 + s_2)$. Thus

$$d(Y, X) \geq d(Y, Z) + d(Z, X) + \log(C) \geq d(Y, \pi(Y)) + d(\pi(Y), X) - (s_1 + s_2) + \log(C).$$

Let $c = s_1 + s_2 - \log(C)$. We get

$$d(Y, X) \geq d(Y, \pi(Y)) + d(\pi(Y), X) - c.$$

If $t < t_Y$, there is a constant s' such that the above holds for g. The claim now follows from the fact that π_f, π_g are uniformly close (see Lemma 184).

□

Definition 205. We call a point $X = (\Gamma, \phi) \in \mathrm{CV}'_n(\mathcal{A})$ *minimal* if Γ is either a relative rose or a graph with two vertices (not special points), one edge between them which we will refer to as a bar, the total of k stems attached to the vertices with the wedge cycles on the other extremities, and all other edges are loops.

Proposition 206. Suppose X is minimal. Let v be one of its vertices (not a special point) and the basepoint for $\pi_1(X, v)$ and let e denote the bar of X initiating from v (e is empty if Γ is a relative rose). Let α_X, β_X be primitive basis elements either one edge loops based at v or loops of the form $e\gamma\bar{e}$ where γ is a one edge loop based at the other vertex. Fix $Z \in \mathcal{L}_f$ and let $h : X \longrightarrow Z$ be a map homotopic to the difference in marking so that $h(\alpha_X)$ is a tight loop and $h(\beta_X)$ is tight as a path. If $h(\alpha_X), [h(\beta_X)]$ both contain a $\jmath$-piece of $\ell_2 \in \Lambda^-$, then $h(\beta_X)$ does not contain a $2\jmath$-piece of $\ell_1 \in \Lambda^+$.

Proof. By Proposition 200, $[h(\beta)]$ does not cross a $\jmath$-piece of ℓ_1 but we want it not to contain any such pieces in the part that gets canceled when we tighten the loop.

We represent $h(\alpha)$ by the edge path x in $\gamma(t)$ and β by $u = wyw^{-1}$, with y cyclically reduced. The paths $x = b_1 a_1 \cdots a_s$ and $u = d_1 c_1 \cdots c_p$, where for any i, a_i, b_i, c_i, d_i are paths in Γ such that $a_i, c_i \subset \bigcup \mathbb{B}_j$ and $b_i, d_i \subset \widehat{\Gamma}$ (a_s and c_p could be empty). Notice that since $\alpha^m\beta$ represents a primitive basis element for all $m \geq 0$, then $x^m u$ represents a primitive basis element for $m \geq 0$. By contradiction, if w crosses an $2\jmath$-piece of ℓ_1 then $w \not\subseteq x^m$, for some $m \geq 1$. Otherwise, x would contain a $\jmath$-piece of ℓ_1 contradicting Proposition 200.

Hence, there is a large enough m such that when we reduce the path $x^m \cdot wyw^{-1} \cdot x^m$ the cancelation happens only at the dots. Write $w = w_1 w_2$ where w_1 is the part that is canceled and $w_2 \neq \emptyset$. If w_2 contains a $\jmath$-piece of ℓ_1, then $z = [x^m \cdot w_1 w_2 y w^{-1} \cdot x^m]$ represents a basis element and w_2 survives after the cancelation. So z will contain a $\jmath$-piece of ℓ_1. If m is large enough, z will also contain a copy of x. We get a primitive basis element containing $\jmath$-pieces of ℓ_1 and ℓ_2 contradicting Proposition 200.

Thus w_1 contains a $\jmath$-piece of ℓ_1. Then x^m contains the inverse of a $\jmath$-piece of ℓ_1 and also a $\jmath$-piece of ℓ_2. Consider the basis element $u = [x^m x^m \cdot wyw^{-1} \cdot x^m]$. The first x^m survives after the cancelation so u contains a $\jmath$-piece of ℓ_2 and the inverse of a $\jmath$-piece of ℓ_1 contradicting Proposition 200. $\square$

Lemma 207. There exist constants $s, c > 0$ such that for $X, Y \in \mathrm{CV}'_n(\mathcal{A})$ if $|t_Y - t_X| > s$, then $d(Y, X) \geq d(Y, \pi(X)) - c$.

Proof. We prove the claim for X, Y such that $t_Y < t_X$. The case where $t_Y > t_X$ follows by applying the same argument to g. First suppose that X is minimal.

Let ℓ_1 be a periodic leaf of Λ_f^+ and ℓ_2 a periodic leaf of Λ_f^-. Let $\jmath$ be as in Proposition 206. The idea of the proof is the following. If $t_Y << t_X$, then for r in the middle of $[t_Y, t_X]$, any loop with positive length corresponding to a primitive element which is short in Y, would contain many $\jmath$-pieces of ℓ_1 in $\gamma(r)$. And any loop with positive length corresponding to a primitive element which is short in X would contain many $\jmath$-pieces of ℓ_2 in $\gamma(r)$. If a candidate in Y was short in X, then it would contain pieces of both ℓ_1 and ℓ_2 in $\gamma(r)$ contradicting the fact that it is a primitive basis element. We need to formalize this argument.

Let s_1 be the constant in Lemma 202, i.e., for any candidate β in Y, $|t_Y - t_\beta| < s_1$. Let s_2 be the constant in Lemma 185, i.e., for any primitive basis element β if $t > t_\beta + s_2$, then $\mathrm{LEG}_f(\beta, \gamma(t)) > \epsilon_0$. Let s_3 be such that if $t > t_\beta + s_2 + s_3$. Then β crosses a $\jmath$-piece of ℓ_1 in $\gamma(t)$ (contributed by one of the κ long legal segments). Let s_4 be such that for any primitive basis element β, if $t < t_\beta - s_4$, then β contains a $\jmath$-piece of ℓ_2 in $\gamma(t)$. Let

$s = 2s_1 + s_2 + s_3 + s_4$ and suppose that $t_X - t_Y > s$ we will show that there exists a c as in the statement of the lemma.

Let β be a loop in Y that realizes $d(Y, \pi(X)) = \log(\mathrm{St}_\beta(Y, \pi(X)))$. By Corollary 202, $t_\beta < t_Y + s_1$. Let $r = t_X - s_1 - s_4$. Then $r > t_Y + s_1 + s_2 + s_3$. Let $k(r)$ be the number of $\jmath$-pieces of ℓ_1 in $\beta_r \subseteq \gamma(r)$ with disjoint interiors, then

$$k(r) \cdot \jmath > \epsilon_0 \cdot l(\beta, \gamma(r)).$$

Recall that X is minimal. Let $\alpha_1, \ldots, \alpha_{n-\sum s(i)}$ denote the classes of minimal length generators of $\pi_1(\widehat{X}, v)$, where α_i and α_j are in the same class if $\widehat{\alpha_i} = \widehat{\alpha_j}$. Hence, α_i is either a one edge loop or is $e\alpha'\bar{e}$ where α' is a one edge loop based at the other vertex and e is the bar of X. Let α_1 be the longest one-edge-loop. Choose a map $h : X \longrightarrow \gamma(r)$, homotopic to the difference in marking, so that $h(\alpha_1)$ is an immersed loop and $h(\alpha_i)$ are immersed as paths. Each $h(\alpha_i)$ in $\gamma(r)$ with positive length contains a $\jmath$-piece of ℓ_2. By Proposition 206, for $1 \leq i \leq n - \sum s(i)$, $h(\alpha_i)$ does not contain any $\jmath$-pieces of ℓ_1.

Claim 208. Let δ be a primitive conjugacy class and write it as a cyclically reduced word in the basis of $\pi_1(X, v)$: $y_1^1, \ldots, y_{s(k)}^k, \alpha_1, \ldots, \alpha_{n-\sum s(i)}$. If $[h(\delta_X)]$ contains k occurrences of $\jmath$-pieces of Λ^+ in $\gamma(r)$ (with disjoint interiors), then δ_X traverses each α_q at least k times.

Proof of Claim. First note that if δ_X is a loop that does not traverse α_q at all then it is carried by the free factor

$$< y_1^1, \ldots, y_k^{s(k)}, \alpha_1, \ldots, \widehat{\alpha_q}, \ldots, \alpha_{n-\sum s(i)} > .$$

Proposition 200 applied to $h(\alpha_q), [h(\delta)]$ in $\gamma(r)$ implies that $[h(\delta)]$ does not contain any $\jmath$-pieces of Λ^+ in $\gamma(r)$. Now suppose that

$$\delta_X = w_1 \alpha_{i_1} w_2 \alpha_{i_2} \ldots w_N \alpha_{i_N} w_{N+1},$$

where $w_i \in \mathcal{A}$ (some w_i's might be trivial) and so that $\alpha_{i_j} = \alpha_q$ for at most $k-1$ choices of j's. We get $\widehat{[h(\delta_X)]} = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_N}$, where σ_{i_j}'s are subpaths of $h(\alpha_{i_1})$ that survives after the cancelation (some σ_{i_j} might be trivial). The $\jmath$-pieces of ℓ_1 can appear only if they are split between different σ_i's. If there are k disjoint $\jmath$-pieces of ℓ_1 in δ, then there is a $\jmath$-piece of ℓ_1 appearing in $\sigma_{i_m} \cdots \sigma_{i_l}$, where $\sigma_{i_j} \subseteq h(\alpha_{i_j})$ and none of the α_{i_j} are equal to α_q. This is a contradiction to the first paragraph. □

By the claim above, β_X in X must traverse α_1 at least $k = k(r)$ times. If $l(\widehat{\alpha_1}, X) > \frac{1}{n - \sum s(i) + 1}$, then $l(\widehat{\beta}, X) > \frac{k}{n - \sum s(i) + 1}$. Otherwise, X has a separating edge e and $l(\widehat{e}, X) >$

$\frac{1}{n-\sum s(i)+1}$. Let δ be a one-edge-loop with positive length so that α_1 and δ are loops on opposite sides of e. By the claim above β_X traverses α_1 and δ alternately at least $\frac{k}{2}$ times therefore it must cross e at least $\frac{k}{2}$ times. Again we get $l(\widehat{\beta}, X) > \frac{k}{2(n-\sum s(i)+1)}$. Therefore,

$$l(\widehat{\beta}, X) > \frac{\epsilon_0}{2(n - \sum s(i) + 1)\jmath} l(\beta, \gamma(r)). \tag{5.3}$$

$\mathcal{L}_f$ is contained in the θ-thick part of $\mathrm{CV}'_n(\mathcal{A})$ for some θ. By Corollary 89 there is a b such that $d(\gamma(t_X), \gamma(r)) < b \cdot d(\gamma(r), \gamma(t_X)) = b(s_1 + s_4)$. Let $\mu = \exp(b(s_1 + s_4))$ then $l(\widehat{\beta}, \gamma(r)) \geq l(\widehat{\beta}, \gamma(t_X))e^{d(\gamma(t_X),\gamma(r))} = \mu l(\widehat{\beta}, \gamma(t_X))$. By equation (5.3) we get $l(\widehat{\beta}, X) > \frac{\epsilon_0 \mu}{2(n-\sum s(i)+1)\jmath} l(\widehat{\beta}, \pi(X))$. Thus, we have $\frac{l(\widehat{\beta},X)}{l(\widehat{\beta},Y)} > \frac{\epsilon_0 \mu}{2(n-\sum s(i)+1)\jmath} \frac{l(\widehat{\beta},\pi(X))}{l(\widehat{\beta},Y)}$, that is,

$$d(Y, X) > d(Y, \pi(X)) - \log\left(\frac{2(n - \sum s(i) + 1)\jmath}{\epsilon_0 \mu}\right).$$

Then the constant c in the statement is $\log\left(\frac{2(n-\sum s(i)+1)\jmath}{\epsilon_0 \mu}\right)$.

Now suppose that X is not minimal. We claim that there is a constant b such that for any $X \in \mathrm{CV}'_n(\mathcal{A})$ there is a minimal G so that $d(X, G) < b$. Moreover, there exists a short loop with positive length in X that is still short in G with positive length. Therefore, by Corollary 203 $d(\pi(G), \pi(X)) < s$. So

$$\begin{aligned} d(Y, X) \geq \quad & d(Y, G) - d(X, G) \geq d(Y, G) - b > d(Y, \pi(G)) - c - b \geq \\ & d(Y, \pi(G)) - d(\pi(G), \pi(X)) - c - b > d(Y, \pi(X)) - c - b - s. \end{aligned}$$

To prove that each point in $\mathrm{CV}'_n(\mathcal{A})$ lies a uniform distance away from a graph G we proceed in the following way. Let e be the longest edge in Γ. Note that $l(\widehat{e}, X) \geq \frac{1}{3n+k-3-3\sum s(i)}$. If e is nonseparating let J be a maximal admissible tree in Γ that does not contain e, where a tree in Γ is admissible if the graph that we get after collapsing the tree is still a marked $(\mathcal{A}, n)$-graph. Otherwise, let J be the forest obtained from this maximal admissible tree by deleting e. Collapse J to get a new unnormalized graph Γ' with volume $> \frac{1}{3n+k-3-3\sum s(i)}$. Notice that Γ' is a minimal graph. Normalize Γ' to get G. Then $d(X, G) \leq \log\left(\frac{1}{1/(3n+k-3-3\sum s(i))}\right) = \log(3n + k - 3 - 3\sum s(i))$. The short basis in X is also short in G. $\square$

Proposition 209. There are constants $s, c > 0$ such that if $d(\pi(Y), \pi(X)) > s$, then

$$d(Y, X) > d(Y, \pi(Y)) + d(\pi(Y), \pi(X)) - c.$$

Proof. By Proposition 204, there are constants c_1 and s_1 so that if $d(\pi(Y),\pi(X)) > s_1$, then

$$d(Y,\pi(X)) > d(Y,\pi(Y)) + d(\pi(Y),\pi(X)) - c_1.$$

By Proposition 207, there are constants c_2 and s_2 so that if $d(\pi(Y),\pi(X)) > s_2$, then

$$d(Y,X) > d(Y,\pi(X)) - c_2.$$

Let $s = \max\{s_1, s_2\}$ and $c = c_1 + c_2$. If $d(Y,X) > s$, then

$$\begin{aligned} d(Y,X) > \quad & d(Y,\pi(X)) - c_2 > d(Y,\pi(Y)) + d(\pi(Y),\pi(X)) - c_1 - c_2 = \\ & d(Y,\pi(Y)) + d(\pi(Y),\pi(X)) - c. \end{aligned}$$

□

As a corollary we get that the projection is coarsely Lipschitz.

Corollary 210. There is a constant c such that for all $X, Y \in \mathrm{CV}'_n(\mathcal{A})$,

$$d(X,Y) \geq d(\pi(X),\pi(Y)) + c.$$

Theorem 211. If $f : \Gamma \longrightarrow \Gamma$ is a train track representative of a fully irreducible relative outer automorphism Φ, then $\mathcal{L}_f$ is D-strongly contracting.

Proof. It is enough to show that there exists a $D > 0$ such that $\mathrm{diam}\{\pi(B_Y(r))\} < D$ for $r = d(Y,\pi(Y))$. We will show that if $X \in B_Y(r)$ then $d(\pi(Y),\pi(X)) < D$, where $D = \max\{s,c\}$ and s and c are the constants in Proposition 209. If $X \in B_Y(r)$, then $d(Y,X) < r$. By Proposition 209, if it is not true that $d(\pi(Y),\pi(X)) \leq s$, then $d(Y,X) > d(Y,\pi(Y)) + d(\pi(Y),\pi(X)) - c$ and

$$r > r + d(\pi(Y),\pi(X)) - c.$$

Thus $d(\pi(Y),\pi(X)) < c$. In both cases $d(\pi(Y),\pi(X)) < \max\{s,c\} < D$. □

Since $\mathcal{L}_f$ is periodic, there is an ϵ so that $\mathrm{Im}\mathcal{L}_f \subseteq \mathrm{CV}'_n(\mathcal{A})^\epsilon$.

Definition 212. We say that $\mathcal{L}_f$ is a *Morse geodesic* if for any (a,b)-quasi-geodesic $\mathcal{Q}$ with endpoints on $\mathcal{L}_f$ there is a C that depends on a, b, ϵ and D so that $d_{\mathrm{Haus}}(\mathrm{Im}\mathcal{L}_f, \mathrm{Im}\mathcal{Q}) < C$.

By Corollary 90 the set $\mathcal{L}_f$ is c-high. Thus applying the Morse Lemma we get the following result.

Theorem 213. $\mathcal{L}_f$ is a Morse geodesic.

In conclusion, first we defined the axes of fully irreducible relative outer automorphisms in the modified relative outer space. Then we defined the projection of this space onto such an axis. Finally, we proved that the axes are strongly contracting Morse geodesics.

CHAPTER 6

APPLICATIONS

We present three applications to the theory developed in the previous chapters. The first application is a Tits alternative for the groups of relative outer automorphisms of free groups. The Tits alternative for $\mathrm{Out}(F_n)$ was proved by Bestvina, Feighn and Handel in [9] and [11]. We give a simple proof of the Tits alternative for $\mathrm{Out}(F_n;\mathcal{A})/\mathrm{KA}$ in the fully irreducible case. The second and most important application is a proof of the non-existence of lattices nontrivially embedded in $\mathrm{Out}(F_n)$. The last application is a study of axes in the Cayley graph of a relative outer automorphism group of a free group modulo the kernel of the action of this group on $\mathrm{CV}'_n(\mathcal{A})$. This application is the relative version of the analog result for $\mathrm{Out}(F_n)$ in [1]. However, in that case the kernel is trivial and the proof rely on the Contracting Geodesics Theorem proved in [1], while our proof is based on the relative version of the Contracting Geodesics Theorem proved in Chapter 5 and the kernel KA could be not trivial.

6.1 A Tits Alternative

A group satisfies the Tits alternative if each of its subgroups either contains a free group of rank two or is virtually solvable. The name Tits alternative comes from the mathematician Jacques Tits who proved in [42] that finitely generated linear groups satisfy this alternative. Ivanov [32] and McCarthy [37] showed this alternative for mapping class groups of compact surfaces. In this case it is also known that solvable subgroups are virtually Abelian by the result of Birman, Lubotzky, and McCarthy [14].

Bestvina, Feighn and Handel proved this alternative for $\mathrm{Out}(F_n)$ in the two papers [9] and [11]. In this section we prove a special case of the Tits alternative for $\mathrm{Out}(F_n;\mathcal{A})/\mathrm{KA}$. The proof of this result is much easier than the proof in the general case.

Theorem 214. Suppose H is a subgroup of $\mathrm{Out}(F_n;\mathcal{A})/\mathrm{KA}$ that contains $[\Phi]$ such that Φ is a fully irreducible relative outer automorphism of infinite order. Then either H contains F_2 or H is virtually cyclic.

Proof. Let $\Lambda^{\pm}$ be the stable and unstable laminations of Φ defined in Chapter 4. We have two cases:

1. H fixes the set $\Lambda^{\pm}$ and so H fixes Λ^{+} or a subgroup of H of index 2 fixes Λ^{+}. Therefore, H is virtually cyclic by Theorem 136.

2. There is $[\rho] \in H$ such that $\rho(\{\Lambda^{\pm}\}) \neq \{\Lambda^{\pm}\}$. If $[\Psi] = [\rho\Phi\rho^{-1}]$, then $\Lambda^{\pm}_{\Psi} = \rho(\Lambda^{\pm})$. By Proposition 151, there exist m, r such that $[\Phi^m]$ and $[\Psi^r]$ generate F_2.

□

6.2 Lattices in $\mathrm{Out}(F_n)$

Let Γ be an irreducible lattice in a semisimple Lie group of real rank at least 2. We are interested in the question: Does every homomorphism $\Gamma \to \mathrm{Out}(F_n)$ have finite image? In this section we will give a positive answer proving the following theorem.

Theorem 215. If Γ is an irreducible lattice in a connected semisimple Lie group of real rank at least 2 with finite center, then every homomorphism $\Gamma \to \mathrm{Out}(F_n)$ has finite image.

The result is known for nonuniform lattices (see [15]). The proof follows easily from Kazdhan-Margulis Finiteness Theorem and the fact that solvable groups of $\mathrm{Out}(F_n)$ are virtually Abelian (see [10]). Theorem 215 was proved by Bridson and Wade, but the proof in the uniform case given here follows the Bestvina-Feighn-Fujiwara approach. In [30], Handel and Mosher proved that if a subgroup H of $\mathrm{Out}(F_n)$ does not contain the class of fully irreducible outer automorphisms, then H has a subgroup of finite index that leaves the conjugacy class of a proper free factor system of F_n invariant.

Definition 216. We say that a free factor system $\mathcal{A}'$ of F_n *properly contains* a free factor system $\mathcal{A}$ of F_n if it contains the (conjugacy classes of) A_i, for $i = 1, \ldots, k$, and there is $A_{k+1} \neq 1$ in $\mathcal{A}'$ such that $F_n = A_1 * \cdots * A_k * A_{k+1} * B'$.

Handel and Mosher proved also the relative version of their theorem.

Lemma 217. For any subgroup $H < \mathrm{Out}(F_n)$ and any proper free factor system $\mathcal{A}$ of F_n, if H fixes the conjugacy class of $\mathcal{A}$, and if no finite index subgroup of H fixes the conjugacy class of any proper free factor system that properly contains $\mathcal{A}$, then there exists $\Phi \in H$ such that no nonzero power of Φ fixes the conjugacy class of any proper free factor system that properly contains the free factor system $\mathcal{A}$.

Bestvina and Feighn [7] showed that if $\Gamma \to \mathrm{Out}(F_n)$ is an embedding, the image does not contain any fully irreducible automorphisms (using the paper of Bestvina and Fujiwara [12]). Hence, the image of the embedding $\varphi : \Gamma \to \mathrm{Out}(F_n)$ is a subgroup $H < \mathrm{Out}(F_n)$ that does not contain any fully irreducible automorphisms.

Recall that a *quasi-homomorphism* on a group G is a function $\phi : G \to \mathbb{R}$ such that

$$\sup\{|\phi(g_1 g_2) - \phi(g_1) - \phi(g_2)| \,|\, g_1, g_2 \in G\} < \infty.$$

Let $\mathrm{QH}(G)$ be the set of quasi-homomorphisms on G. Notice that $\mathrm{QH}(G)$ is a vector space. Let $\widetilde{\mathrm{QH}}(G)$ be the quotient of $\mathrm{QH}(G)$ by bounded functions and homomorphisms $G \to \mathbb{R}$. We can think of $\widetilde{\mathrm{QH}}(G)$ as the kernel of the natural homomorphism from the second bounded cohomology of G with real coefficients to the standard second cohomology of G with real coefficients. We will prove the following lemma.

Lemma 218. If $\mathrm{Out}(F_n; \mathcal{A})$ contains at least one exponentially growing element, then the dimension of $\widetilde{\mathrm{QH}}(\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA})$ is infinite. Moreover, if $H < \mathrm{Out}(F_n)$ is finitely generated, not virtually Abelian, and it does not contain fully irreducible elements, then there exists a finite index subgroup $H_1 < H$ such that $H_1 < \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ and if H_1 contains at least one exponentially growing element, then $\dim(\widetilde{\mathrm{QH}}(H_1)) = \infty$.

In order to prove Lemma 218 we will introduce the projection complex for the group $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ such that $\mathrm{Out}(F_n; \mathcal{A})$ contains at least one exponentially growing element.

6.2.1 Projection Complex for $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$

We will recall the definition of the projection complex given in [5]. Let Y be a set and assume that for each $y \in Y$ we have a function

$$d_y^\pi : (Y \setminus \{y\}) \times (Y \setminus \{y\}) \to [0, \infty)$$

and a constant $c > 0$ such that the following axioms are satisfied:

1. $d_y^\pi(x, z) = d_y^\pi(z, x)$;
2. $d_y^\pi(x, w) \leq d_y^\pi(x, z) + d_y^\pi(z, w)$;
3. $\min\{d_y^\pi(x, z), d_z^\pi(x, y)\} < c$;
4. $|\{y | d_y^\pi(x, z) \geq c\}|$ is finite for all $x, z \in Y$.

Consider the modified relative outer space $\mathrm{CV}'_n(\mathcal{A})$ equipped with the Lipschitz metric defined in Chapter 3. Fully irreducible relative outer automorphisms in $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$

have axes in $\mathrm{CV}'_n(\mathcal{A})$ as constructed in Chapter 5. Let $\Phi_1, \ldots, \Phi_l$ be a finite collection of fully irreducible relative outer automorphisms and let $\gamma_1, \ldots, \gamma_l$ be their axes.

Take Y to be the set of parallel classes of $(\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA})$-translates of the γ_i's, where two lines are parallel if each is contained in a Hausdorff neighborhood of the other. By Theorem 213, there is a constant $C > 0$ such that the projection of any translate $\gamma_i \cdot \Phi$ to any nonparallel γ_j is bounded by C.

Notice that this is equivalent to the *Weak Proper Discontinuity Condition* defined in [12]. Define $d_y^{\pi}(x, z) = \operatorname{diam}(\pi_\gamma(\alpha \cup \beta))$, for any $\alpha \in x$, $\beta \in z$, $\gamma \in y$. It is easy to check that all axioms hold. Now define $d_Y(x, z) = \inf_{(x', z') \in \mathcal{H}(x,z)} d_y^{\pi}(x, z)$, where $\mathcal{H}(x, z)$ is the set of $(x', z') \in Y \times Y$ such that one of the following holds:

- $d_x^{\pi}(x', z')$, $d_z^{\pi}(x', z') > 2\xi$;
- $x = x'$ and $d_z^{\pi}(x, z') > 2\xi$;
- $z = z'$ and $d_x^{\pi}(x', z) > 2\xi$;
- $(x', z') = (x, z)$.

where $\xi > 0$ is a constant.

Let $Y_K(x, z)$ be the set of all $y \in Y$ such that $d_Y(x, z) > K$. We define the 1-complex $\mathcal{P}_K(Y)$ as follows:

- the vertices are elements in Y;
- there is an edge between two vertices x and z if $Y_K(x, z)$ is empty.

In [5], Bestvina, Bromberg and Fujiwara proved the following theorem.

Theorem 219. For K big enough, $\mathcal{P}_K(Y)$ is connected and it is quasi-isometric to a simplicial tree.

Therefore, there exists a hyperbolic space on which $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ acts. We will denote by X such space.

6.2.2 Proof of Lemma 218

First we recall definitions and results in [12]. Let G be a discrete group acting by isometry on a space X. An isometry g of X is hyperbolic if it admits an invariant quasi-geodesic. For example, a fully irreducible element of $\mathrm{Out}(F_n; \mathcal{A})$ is hyperbolic. We say that two hyperbolic elements $g_1, g_2 \in G$ are $g_1 \sim g_2$ if for any long segment L of the invariant quasi-geodesic there exists a $g \in G$ so that $g(L)$ is in a δ-Hausdorff neighborhood of the invariant quasi-geodesic of g_2 and $L \to g(L)$ is orientation-preserving with respect the orientation of g_1 on L and on g_2 on $g(L)$.

Proposition 220 ([12] Proposition 6)**.** If an action of a group G on a space X satisfies the Weak Proper Discontinuity Condition, then there exist hyperbolic $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$.

Theorem 221 ([12] Theorem 1)**.** Suppose that G is a group acting on a δ-hyperbolic graph by isometries and there exist independent hyperbolic $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$. Then the dimension of $\widetilde{QH}(G)$ is infinite.

Theorem 222 ([12] Theorem 7)**.** Suppose that an action of a group G on a space X satisfies the Weak Proper Discontinuity Condition. Then the dimension of $\widetilde{QH}(G)$ is infinite.

The proof the first statement of Lemma 218 is a consequence of Theorem 219 and Proposition 6 and Theorem 1 in [12]. Indeed, by Theorem 219 there exists a hyperbolic space X on which $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ acts and it satisfies the Weak Proper Discontinuity Condition. Combining Proposition 6 and Theorem 1 in [12] the first part of the proposition follows. In order to prove the second part, because H does not contain any fully irreducible outer automorphism, let $\mathcal{A}$ be the biggest free factor system preserved by H. By the result of Handel and Mosher there exists a finite index subgroup $H_1 < H$ such that $\mathcal{A}$ is invariant under H_1. We have the following cases:

1. H_1 contains two independent fully irreducible relative outer automorphisms.
2. H_1 fixes a pair $\Lambda^{\pm}$ of laminations corresponding to the fully irreducible relative outer automorphisms. Then $H_1 < \mathrm{Stab}(\Lambda^{\pm})$ is virtually cyclic.
3. H_1 is finite.

Since H is not virtually Abelian, we must exclude case (2) and case (3). Then the action of H_1 on X satisfies the assumption of Theorem 7 in [12] and this concludes the proof of Lemma 218. Note that a similar statement is true if we replace $\mathrm{Out}(F_n)$ by $\mathrm{Out}(F_n; \mathcal{A})$.

6.2.3 Proof of Theorem 215

In order to prove Theorem 215, we will need the following lemma, which is a consequence of Margulis super-rigidity:

Lemma 223. If Γ is an irreducible lattice in a connected semisimple Lie group of real rank at least 2, and Γ is embedded in a group G such that

$$1 \to A \to G \to B \to 1$$

is an exact sequence, then $\Gamma' \hookrightarrow A$ or $\Gamma' \hookrightarrow B$, where $\Gamma' < \Gamma$ is of finite index or $\Gamma' = \Gamma/\mathrm{Ker}(\Gamma \hookrightarrow G)$ and $\mathrm{Ker}(\Gamma \hookrightarrow G)$ is finite respectively.

Remark 224. Γ' is a lattice in a (maybe different) higher rank Lie group.

We know that if $\varphi : \Gamma \to \mathrm{Out}(F_n)$ is an embedding, the image H does not contain any fully irreducible outer automorphisms. Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be the biggest free factor system preserved by H. Consider the exact sequence

$$1 \to K \to H \to \mathrm{Out}(A_1) \times \cdots \times \mathrm{Out}(A_k) \to 1$$

and notice that $K < \mathrm{Out}(F_n; \mathcal{A})$. By Lemma 223 there are two cases:

- $\Gamma' \hookrightarrow \mathrm{Out}(A_1) \times \cdots \times \mathrm{Out}(A_k)$ and in this case we can proceed by induction considering

$$1 \to \mathrm{Out}(A_2) \times \cdots \times \mathrm{Out}(A_k) \xrightarrow{i} \mathrm{Out}(A_1) \times \cdots \times \mathrm{Out}(A_k) \xrightarrow{p_1} \mathrm{Out}(A_1) \to 1,$$

  where i is the inclusion and p_1 is the projection onto the first factor.

- $\Gamma' \hookrightarrow K < \mathrm{Out}(F_n; \mathcal{A})$ and in this case let H_1 be the image of $\Gamma' \hookrightarrow K$.

In the latter case we have two different subcases: H_1 contains at least one exponentially growing element or H_1 does not contain any exponentially growing elements. If H_1 does not contain any exponentially growing elements, then there are two cases (see [9]):

1. $\sum_{A_i \in \mathcal{A}} \mathrm{rk}(A_i) = n$ and $k \leq 2$ or

2. $\sum_{A_i \in \mathcal{A}} \mathrm{rk}(A_i) = n - 1$ and $k = 1$.

In the first case, by Stalling's method an element in $\mathrm{Out}(F_n;\mathcal{A})$ is of the form

$$\theta^i \mapsto \omega_i(A_i)\theta^i\overline{\omega_i}(A_i),$$

where θ^i is the $s(i)$-plet of generators of A_i, and $\omega(A_i) \in A_i$. Hence, $\mathrm{Out}(F_n;\mathcal{A}) \cong A_1 \times \cdots \times A_k$. In the second case, by Stalling's method an element in $\mathrm{Out}(F_n;\mathcal{A})$ is of the form

$$\begin{array}{ccc} \theta^1 & \mapsto & \omega_1(A_1)\theta^1\overline{\omega_1}(A_1) \\ b & \mapsto & u_1 b u_2 \end{array}$$

where θ^1 is the $(n-1)$-plet of generators of A_1, b is the generator of B, $\omega(A_1) \in A_1$, and $u_1, u_2 \in \mathcal{A} * B$. Hence, $\mathrm{Out}(F_n;\mathcal{A}) \cong A_1 \times F_{n-1} \times F_{n-1}$. In any of the two cases we can consider the sequence

$$1 \to K \to H_1 \to A_1 \to 1,$$

where K is the kernel of the projection of the group onto A_1. By Lemma 223 and by induction on the free factor system we reduce to the case $\Gamma \hookrightarrow F_k$, for some free group F_k. Therefore, the image of $\Gamma \to \mathrm{Out}(F_n)$ is finite. Now, suppose that H_1 contains at least one exponentially growing element. Consider

$$1 \to \mathrm{KA} \to \mathrm{Out}(F_n;\mathcal{A}) \to \mathrm{Out}(F_n;\mathcal{A})/\mathrm{KA} \to 1.$$

Again by Lemma 223 we have two options: $\Gamma'' < \mathrm{KA} \cong A_1 \times \cdots \times A_k$ (see Corollary 51 for the isomorphism) and we run an induction on the A_i's, or $\varphi_1 : \Gamma'' \hookrightarrow H_1' < \mathrm{Out}(F_n;\mathcal{A})/\mathrm{KA}$ and H_1' contains at least one exponentially growing element. In the latter case, since $\dim(\widetilde{\mathrm{QH}}(H_1')) = \infty$ (by Lemma 218), $\mathrm{Im}(\varphi_1)$ does not contain any fully irreducible relative outer automorphism in $\mathrm{Out}(F_n;\mathcal{A})$. Indeed, Burger and Monod proved that $\widetilde{\mathrm{QH}}(\Gamma) = 0$ (see [17]). Therefore, the image of $\varphi_1 : \Gamma'' \to H_1'$ does not contain more than one fully irreducible relative outer automorphism. Suppose that $\Phi \in H_1'$ is fully irreducible. If H_1' leaves $T_\Phi^\pm$ invariant, then H_1' and Γ'' are virtually cyclic by Theorem 136 and this is not possible. On the other hand, if there exists $\Psi \in H_1'$ that does not preserve $T_\Phi^\pm$, then Φ and $\Psi\Phi\Psi^{-1}$ are independent fully irreducible relative outer automorphisms in H_1' and this contradicts the previous observation. In any case we conclude that $\varphi_1 : \Gamma'' \to H_1'$ does not contain any fully irreducible relative outer automorphisms. Lemma 223, Lemma 218, and Lemma 217 allow us to run an inductive argument on the rank of the invariant free factor systems of F_n. Since any induction has to stop in a finite amount of steps, we necessarily have that every homomorphism $\Gamma \to \mathrm{Out}(F_n)$ has finite image.

6.3 Axes in the Cayley graph of $\mathrm{Out}(F_n; \mathcal{A})$

Recall that KA is the kernel of the action of $\mathrm{Out}(F_n; \mathcal{A})$ on $\mathrm{CV}'_n(\mathcal{A})$ described in Section 3.1.3. Let $\mathcal{C}$ be the Cayley graph of $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$, with the class of Nielsen-Whitehead generators $\{\Phi_i\}_{i=1}^N$ defined in Corollary 97 that fix the generators of $\mathcal{A}$, that is, $\mathcal{C}$ has a vertex for each element in $\mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$ and two vertices $[\Psi_1]$ and $[\Psi_2]$ are connected by an edge if there is a Nielsen-Whitehead generator Φ_i such that $\Psi_1 = \Phi_i \circ \Psi_2$ modulo KA. Let Φ be a fully irreducible relative outer automorphism. Let $f : \Gamma \longrightarrow \Gamma$ be a train track map for Φ and let λ be the expansion factor for Φ. Choose an embedding $\iota : \mathcal{C} \hookrightarrow \mathrm{CV}'_n(\mathcal{A})$ as follows. Let L be the axis for $[\Phi]$ in $\mathcal{C}$. Consider the vertex $[\Phi] \in L$ and map $\iota(\Phi) = X = (\Gamma, \phi)$, where the marking ϕ is induced by Φ. Extend ι to the vertices of $\mathcal{C}$ equivariantly and to the edges of $\mathcal{C}$ by mapping them onto some geodesic between the images of their endpoints. Following [1] our goal is to prove that L is a Morse geodesic in $\mathcal{C}$.

We will denote by $d_{\mathrm{CV}'_n(\mathcal{A})}(\cdot,\cdot)$ the Lipschitz distance in the modified relative outer space $\mathrm{CV}'_n(\mathcal{A})$ and by $d_{\mathcal{C}}(\cdot,\cdot)$ the distance in the Cayley graph $\mathcal{C}$. Note that if $[\Psi_1], [\Psi_2] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$, then $d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Psi_1]), \iota([\Psi_2]))$ is well defined. Let

$$M = \max\{d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Phi_i]), \iota([\mathrm{id}]))|[\Phi_i] \text{ is a generator}\}.$$

For $[\Psi_1], [\Psi_2] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$, we have

$$d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Psi_1]), \iota([\Psi_2])) \leq M \cdot d_{\mathcal{C}}([\Psi_1], [\Psi_2]).$$

However, this is not a quasi-isometric embedding, since this is not even true for $\mathrm{Out}(F_n)$ (see [1]). As in the case of the standard outer space, for points on L, distances in $\mathrm{CV}'_n(\mathcal{A})$ coarsely correspond to distances in $\mathcal{C}$. Indeed, for $m > 0$, $d_{\mathrm{CV}'_n(\mathcal{A})}(X, X \cdot \Phi^m) = m \log(\lambda)$ (see Corollary 76). Let $|\Phi|_{\mathcal{C}}$ be the translation length of Φ, and $[\Psi] \in L$, then

$$d_{\mathcal{C}}([\Psi], [\Psi\Phi^m]) = m \cdot |\Phi|_{\mathcal{C}} = \frac{d_{\mathrm{CV}'_n(\mathcal{A})}(X, X \cdot \Phi^m)}{\log(\lambda)} |\Phi|_{\mathcal{C}} =$$

$$= \frac{|\Phi|_{\mathcal{C}}}{\log(\lambda)} d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Psi]), \iota([\Psi\Phi^m])).$$

Since L is mapped into the thick part, by Corollary 90 we have

$$d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Phi^m\Psi]), \iota([\Psi])) \geq \frac{1}{A} d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Psi]), \iota([\Psi\Phi^m])) =$$

$$= \frac{\log(\lambda)}{A|\Phi|_{\mathcal{C}}} d_{\mathcal{C}}([\Psi], [\Psi\Phi^m]) = d_{\mathcal{C}}([\Phi^m\Psi], [\Psi]),$$

for some A.

In order to prove that L is a Morse geodesic we need the following lemma.

Lemma 225. For every $a > 0$ there is a $b > 0$ such that if $d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Psi]), \iota([\Upsilon])) < a$, then $d_{\mathcal{C}}([\Psi], [\Upsilon]) < b$, for any $[\Psi], [\Upsilon] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$.

Proof. The image of ι is discrete, hence there are finitely many $d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\mathrm{id}]), \iota([\Theta])) < a$ such that $[\Theta] \in \mathrm{Out}(F_n; \mathcal{A})/\mathrm{KA}$. Let

$$b = \max\{d_{\mathcal{C}}([\mathrm{id}], [\Theta]) \mid d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\mathrm{id}]), \iota([\Theta])) < a\}.$$

Suppose $d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\Psi]), \iota([\Upsilon])) < a$. Then

$$d_{\mathrm{CV}'_n(\mathcal{A})}(\iota([\mathrm{id}]), \iota([\Upsilon\Psi^{-1}])) < a,$$

so $d_{\mathcal{C}}([\Psi], [\Upsilon]) = d_{\mathcal{C}}([\mathrm{id}], [\Upsilon\Psi^{-1}]) < b$. □

Theorem 226. *L is a Morse geodesic in $\mathcal{C}$.*

Proof. Let α be an (a, b)-quasi geodesic (see Definition 190) in $\mathcal{C}$ with endpoints on L. By Lemma 192, we may assume that α is tame. Consider $\gamma = \iota \circ \alpha$, then the length of $\gamma|_{[t,t']}$ is smaller or equal to $M \cdot \mathrm{len}_{\mathcal{C}}(\alpha|_{[t,t']}) \leq Ma(t - t') + Mb$. By Remark 196, there exists a constant $d = d(a, b, M, D, \varepsilon)$, where $\mathcal{L}_f \subset \mathrm{CV}'_n(\mathcal{A})^{\varepsilon}$ such that $d_{\mathrm{Haus}}(\mathrm{Im}\gamma, \mathrm{Im}\mathcal{L}_f) < d$. By Lemma 225, we have $d_{\mathrm{Haus}}(\mathrm{Im}L, \mathrm{Im}\alpha) < B$, for some $B = B(d)$. □

In this chapter we gave three applications of the theory about relative outer automorphisms developed in the previous chapters. The second and most interesting application shows how to construct an inductive argument making use of the relative outer automorphisms. A similar technique can be utilize in solving other interesting problems.

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